

Character correspondences induced by magic representations

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ABSTRACT. Let G be a finite group, K a normal subgroup of G and $H \leq G$ such that $G = HK$, and set $L = H \cap K$. Suppose $\vartheta \in \text{Irr } K$ and $\varphi \in \text{Irr } L$, and φ occurs in ϑ_L with multiplicity $n > 0$. A projective representation of degree n on H/L is defined in this situation; if this representation is ordinary, it yields a Morita equivalence between $\mathbb{C}Ge_\vartheta$ and $\mathbb{C}He_\varphi$, and thus a bijection between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$. The behavior of fields of values and Schur indices under this bijection is described. A modular version of the main result is proved. We show that the theory applies if n and $|H/L|$ are coprime. Finally, assume that $P \leq G$ is a p -group with $P \cap K = 1$ and PK normal in G , that $H = \mathbf{N}_G(P)$, and that ϑ and φ belong to blocks of p -defect zero which are Brauer correspondents with respect to the group P . Then every block of $\mathbb{F}_p G$ or $\mathbb{Q}_p G$ lying over ϑ is Morita-equivalent to its Brauer correspondent with respect to P . This strengthens a result of Turull [*Above the Glauberman correspondence*, *Advances in Math.* **217** (2008), 2170–2205].

1. INTRODUCTION

Let G be a finite group, $K \trianglelefteq G$ and $H \leq G$ such that $G = HK$, and set $L = H \cap K$. Suppose $\vartheta \in \text{Irr } K$ and $\varphi \in \text{Irr } L$ are invariant in G respective H . We wish to compare the sets of characters $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$. Questions of this type occur in many applications, and so this situation has already been studied by many authors [3, 6, 11, 13, 16, 30]. In particular, there are many results that construct, in special cases of the above situation, a bijection between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$.

Here we will assume that $(\vartheta_L, \varphi) = n > 0$, that is, φ occurs in the restriction of ϑ to L with non-zero multiplicity n . In this situation, an n -dimensional projective representation of H/L arises naturally, as we will see. If this representation turns out to be projectively equivalent to an ordinary, “honest” representation, it yields a bijection between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$. This establishes a new, quite general approach for finding such a character correspondence.

Let us give a somewhat more detailed outline of the idea. Let $e_\vartheta \in \mathbb{C}K$ and $e_\varphi \in \mathbb{C}L$ be the central primitive idempotents associated with the characters ϑ and φ and set $i = e_\vartheta e_\varphi$. This is a nonzero idempotent in $\mathbb{C}K$. Let $S = (i\mathbb{C}Ki)^L = \mathbf{C}_{i\mathbb{C}Ki}(L)$, the centralizer of L in $i\mathbb{C}Ki$. It can be shown that $S \cong \mathbf{M}_n(\mathbb{C})$, an $n \times n$ -matrix ring over \mathbb{C} . The group H/L acts on S by conjugation, and for every $Lh \in H/L$ there is $\sigma(Lh) \in S^*$ such that $s^h = s^{\sigma(Lh)}$ for all $s \in S$. This defines a projective representation $\sigma: H/L \rightarrow S$. If σ is multiplicative, we call it a “magic” representation. We will show that every magic representation σ determines uniquely a bijection

$$\iota = \iota(\sigma): \text{Irr}(G \mid \vartheta) \rightarrow \text{Irr}(H \mid \varphi)$$

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with good compatibility properties (see Theorem 4.3 for the precise statement). A property particular to our situation is the following: Let ψ be the character of the magic representation σ . Then for $\chi \in \text{Irr}(G \mid \vartheta)$ we have

$$\sum_{\xi \in \text{Irr}(H \mid \varphi)} (\chi_H, \xi) \xi = \psi \chi^\ell.$$

Note that the left hand side is the part of χ_H that lies above φ .

As most readers probably know, the G -invariant character ϑ determines a cohomology class of G/K , that is, an element of $H^2(G/K, \mathbb{C}^*)$. We denote this element by $[\vartheta]_{G/K}$. Similarly, φ determines a cohomology class $[\varphi]_{H/L} \in H^2(H/L, \mathbb{C}^*)$. As $G/K \cong H/L$ canonically, it is meaningful to compare $[\vartheta]_{G/K}$ and $[\varphi]_{H/L}$. One can show that $[\vartheta]_{G/K} [\varphi]_{H/L}^{-1}$ is just the cohomology class belonging to the projective representation σ described above. This also explains the existence of the character bijection, since it is known that $[\vartheta]_{G/K}$ determines $\text{Irr}(G \mid \vartheta)$, in some sense [15, Chapter 11], and similarly for φ .

The results described so far are proved in Section 3 and 4. In fact, we work with a field \mathbb{F} containing the values of ϑ and φ instead of \mathbb{C} . We show also that our correspondence has good rationality properties.

In Section 6, we go one step further and skip the assumption that ϑ and φ are invariant in H . This can not be handled simply by the well known Clifford correspondence between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(G_\vartheta \mid \vartheta)$ for the following reason: Suppose $\xi \in \text{Irr}(G_\vartheta \mid \vartheta)$ and $\chi = \xi^G$ is its Clifford correspondent. Then $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\xi)$, but this may be a strict containment. In particular, we may have $\mathbb{Q}(\vartheta) \not\subseteq \mathbb{Q}(\chi)$. Of course $\mathbb{Q}(\chi_K) \subseteq \mathbb{Q}(\chi)$, and $\mathbb{Q}(\chi_K)$ depends only on ϑ and the embedding of K in G , but not on χ itself. Since we want results as general as possible about the behavior of fields of values and Schur indices under our character correspondences, we have to work over a field not necessarily containing the values of ϑ and φ . This involves various technical difficulties. Some preliminary material is contained in Section 5.

In Section 7 we show that the results apply when $n = (\vartheta_L, \varphi)$ and $|H/L|$ are coprime. This slightly generalizes a result of Schmid [27].

In Section 8 we prove a technical result that can be used for inductive proofs, and in Sections 9 and 10 we study relations with modular representation theory.

Finally in Section 11 we assume that, additionally, we have a p -subgroup $P \leq G$ such that $KP \trianglelefteq G$ and $P \cap K = 1$, that ϑ and φ have p -defect zero (in K respectively in L) and that their blocks are Brauer correspondents with respect to P , and we let $H = \mathbf{N}_G(P)$. (If K happens to be a p' -group, then ϑ and φ are Glauberman correspondents.) We show that our theory applies in this situation, thus getting Morita equivalences between group algebras over the prime field with p elements and over the field of p -adic numbers (Corollaries 11.2 and 11.3). In fact, we even get Morita equivalences between blocks over ϑ and φ which are Brauer correspondents with respect to P . This generalizes a recent result of Turull [30] and provides an alternative proof thereof.

The theory of magic representations also applies above fully ramified sections of a group, as studied by Isaacs [11] and others. Indeed, much more information on the character of the magic representation is known in this case. In particular, in another paper [18] we use the methods of the present paper to show that a character correspondence described by Isaacs preserves Schur indices over the rational numbers.

The results of this paper are part of my doctoral thesis [17], but some of them appear here in a more general form than in my thesis.

2. CENTRAL FORMS AND CENTRAL SIMPLE SUBALGEBRAS

We review some preliminary material in this section that we need later. Let C be a ring. We write $\mathbf{M}_n(C)$ to denote the ring of $n \times n$ -matrices with entries in C . Let E_{ij} be the matrix such that the entry with index (i, j) is 1_C , and all the other entries are 0_C . The set $\{E_{ij} \mid i, j = 1, \dots, n\}$ has the following properties:

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \quad \text{and} \quad \sum_{i=1}^n E_{ii} = 1.$$

Now let A be any ring. Any subset $E = \{E_{ij} \mid i, j = 1, \dots, n\}$ of A with these properties is called a full set of matrix units in A . If such a full set of matrix units exists, then $A \cong \mathbf{M}_n(C)$, where $C = \mathbf{C}_A(E)$ [19, pp. 17.4-17.6]. The isomorphism sends the $n \times n$ -matrix (c_{ij}) to $\sum_{i,j} c_{ij}E_{ij}$.

Now suppose that A is an R -algebra, where R is some commutative ring, and $E \subseteq A$ is a full set of matrix units. Let S be the R -subalgebra generated by E . Then, by the result cited above, we have $S \cong \mathbf{M}_n(R)$. It is also clear that $\mathbf{C}_A(S) = \mathbf{C}_A(E) =: C$. It follows that $A \cong S \otimes_R C$ canonically, via $s \otimes c \mapsto sc$.

In the next few results, we will consider the following situation: Let \mathbb{F} be a field and S a central simple \mathbb{F} -algebra. Suppose S is a subalgebra of the \mathbb{F} -algebra A and set $C = \mathbf{C}_A(S)$. The following lemma is probably well known:

2.1. Lemma. $S \otimes_{\mathbb{F}} C \cong A$ canonically (via $s \otimes c \mapsto sc$).

Proof. Define an algebra homomorphism $\kappa: S \otimes_{\mathbb{F}} C \rightarrow A$ by $(s \otimes c)^{\kappa} = sc$. We have seen above that κ is an isomorphism if $S \cong \mathbf{M}_n(\mathbb{F})$. In the general case, there is a field $\mathbb{E} \geq \mathbb{F}$ such that $S \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbf{M}_n(\mathbb{E})$. Then $\mathbf{C}_{A \otimes_{\mathbb{F}} \mathbb{E}}(S \otimes_{\mathbb{F}} \mathbb{E}) = C \otimes_{\mathbb{F}} \mathbb{E}$, and $(S \otimes_{\mathbb{F}} C) \otimes_{\mathbb{F}} \mathbb{E} \cong (S \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (C \otimes_{\mathbb{F}} \mathbb{E})$. It follows that $\kappa \otimes 1: (S \otimes_{\mathbb{F}} C) \otimes_{\mathbb{F}} \mathbb{E} \rightarrow A \otimes_{\mathbb{F}} \mathbb{E}$ is an isomorphism, and thus κ is also an isomorphism. \square

2.2. Notation. Let R be a commutative ring and A an R -algebra. We denote by $\text{ZF}(A, R)$ the set of central R -forms on A , that is the set of R -linear maps $\tau: A \rightarrow R$ with $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

Note that an R -linear map $\tau: A \rightarrow R$ is a central form if and only if $[A, A] = \{\sum_i (a_i b_i - b_i a_i)\} \subseteq \ker \tau$. Using this, one easily proves:

2.3. Lemma. Let A and B be R -algebras. Then

$$(\sigma, \tau) \mapsto \sigma \otimes \tau \in \text{ZF}(A \otimes_R B, R), \quad (\sigma \otimes \tau)(a \otimes b) = \sigma(a)\tau(b)$$

defines a canonical isomorphism

$$\text{ZF}(A, R) \otimes_R \text{ZF}(B, R) \cong \text{ZF}(A \otimes_R B, R).$$

We return to the situation where S is a central simple algebra over the field \mathbb{F} . We denote the reduced trace of the central simple \mathbb{F} -algebra S by $\text{tr}_{S/\mathbb{F}}$ or simply tr , if no confusion can arise. Remember that the reduced trace is computed as follows: first choose a splitting field \mathbb{E} of S and an isomorphism $\varepsilon: S \otimes_{\mathbb{F}} \mathbb{E} \rightarrow \mathbf{M}_n(\mathbb{E})$, then let $\text{tr}_{S/\mathbb{F}}(s)$ be the ordinary matrix trace of $(s \otimes 1)^{\varepsilon}$. Then indeed $\text{tr}_{S/\mathbb{F}}(s)$ is a well-defined element of \mathbb{F} [25, Section 9a]. The kernel of $\text{tr}_{S/\mathbb{F}}$ is exactly the subspace $[S, S]$. Thus $\text{ZF}(S, \mathbb{F}) = \mathbb{F} \cdot \text{tr}_{S/\mathbb{F}} \cong \mathbb{F}$.

Now suppose $S \leq A$ and $C = \mathbf{C}_A(S)$ as above. Combining these remarks with Lemma 2.1 and Lemma 2.3, we get that

$$\text{ZF}(C, \mathbb{F}) \cong \text{ZF}(S, \mathbb{F}) \otimes_{\mathbb{F}} \text{ZF}(C, \mathbb{F}) \cong \text{ZF}(S \otimes_{\mathbb{F}} C, \mathbb{F}) \cong \text{ZF}(A, \mathbb{F}).$$

Any central form $\chi \in \text{ZF}(A, \mathbb{F})$ can be written as $\text{tr}_{S/\mathbb{F}} \otimes \tau$ for some $\tau \in \text{ZF}(C, \mathbb{F})$. The next lemma describes τ in terms of χ .

2.4. Lemma. *Pick $s_0 \in S$ with $\text{tr}_{S/\mathbb{F}}(s_0) = 1$. Consider*

$$\text{ZF}(A, \mathbb{F}) \ni \chi \mapsto \chi^\varepsilon \in \text{ZF}(C, \mathbb{F}), \quad \chi^\varepsilon(c) = \chi(s_0 c) \quad \text{for } c \in C.$$

Then ε is an isomorphism, is independent of the choice of s_0 and

$$\chi(sc) = \text{tr}_{S/\mathbb{F}}(s) \chi^\varepsilon(c) \quad \text{for all } s \in S \text{ and } c \in C.$$

Proof. We have $[S, S]C = [SC, S] \subseteq [A, A] \subseteq \ker \chi$ for any $\chi \in \text{ZF}(A, \mathbb{F})$. Since $\dim_{\mathbb{F}}(S/[S, S]) = 1$, it follows that for any $s \in S$ we have $s - \text{tr}_{S/\mathbb{F}}(s)s_0 \in [S, S]$. Thus

$$\chi(sc) = \chi((s - \text{tr}_{S/\mathbb{F}}(s)s_0)c + \text{tr}_{S/\mathbb{F}}(s)s_0 c) = \text{tr}_{S/\mathbb{F}}(s) \chi(s_0 c).$$

This shows the last claim. If $\tau \in \text{ZF}(C, \mathbb{F})$, then $\hat{\tau}(sc) = \text{tr}_{S/\mathbb{F}}(s) \tau(c)$ defines a central form $\hat{\tau}$ on A , since $A \cong S \otimes_{\mathbb{F}} C$. It is clear now that $\tau \mapsto \hat{\tau}$ is the inverse of ε . Since the definition of $\hat{\tau}$ is independent of s_0 , the map ε is independent of s_0 , too. The proof is finished. \square

2.5. Lemma. *Assume that $S \cong \mathbf{M}_n(\mathbb{F})$. Then χ^ε affords a C -module if and only if χ affords an A -module.*

Proof. Identify A with $S \otimes C$. Let V be a simple S -module. Then $\text{tr}_{S/\mathbb{F}} = \text{tr}_V$. Thus if χ^ε is the character of the C -module M , then χ is the character of the A -module $V \otimes_{\mathbb{F}} M$. Conversely, suppose χ is the character of the $S \otimes_{\mathbb{F}} C$ -module N . Since S is split, there is an idempotent $e \in S$ of trace 1. Then the character of Ne as C -module is obviously χ^ε . \square

Remark. If S is possibly not split and V a simple S -module, then $\text{tr}_V = m \text{tr}_{S/\mathbb{F}}$ for some $m \in \mathbb{N}$ (the Schur index of S). Thus if χ^ε affords the C -module M , then $m\chi$ affords the A -module $V \otimes M$.

2.6. Remark. The proofs of the last lemmas are easier when $S \cong \mathbf{M}_n(\mathbb{F})$. Moreover, in this case Lemmas 2.1, 2.4 and 2.5 remain true for commutative rings \mathbb{F} , not just fields.

Now let R be a commutative ring and A an R -algebra. An idempotent $i \in A$ is called a *full idempotent* of A if $AiA = A$. It is well known that A and iAi are Morita equivalent when i is full [19, p. 18.30]. The following is also well known.

2.7. Lemma. *Let A be an R -algebra and i a full idempotent of A . Set $C = iAi$. Then restriction to C defines an isomorphism from $\text{ZF}(A, R)$ onto $\text{ZF}(C, R)$, and $\tau|_C$ is the character of a C -module if and only if τ is the character of an A -module.*

Proof. Suppose $1_A = \sum_{k=1}^r x_k i y_k$ with $x_k, y_k \in A$. If $\xi \in \text{ZF}(C, R)$, then $\hat{\xi}$ defined by $\hat{\xi}(b) = \sum_{k=1}^r \xi(i y_k b x_k i)$ is a central form, as a routine calculation shows. The map $\xi \mapsto \hat{\xi}$ is the inverse of restriction.

It is clear that if χ is the character of the A -module M , then $\chi|_C$ is the character of Mi as C -module. Conversely if $\chi|_C$ is the character of the C -module N , then χ is the character of the A -module $N \otimes_C iA$. \square

3. MAGIC REPRESENTATIONS

Throughout this section we assume the following situation:

3.1. Hypothesis. Suppose $G = HK$ is a finite group where $K \trianglelefteq G$ and $H \leq G$, and set $L = H \cap K$. Let \mathbb{F} be a field with algebraic closure $\overline{\mathbb{F}}$. Let $\vartheta \in \text{Irr}_{\overline{\mathbb{F}}} K$ and $\varphi \in \text{Irr}_{\overline{\mathbb{F}}} L$ be irreducible characters, such that φ occurs in ϑ_L with multiplicity $n > 0$. Assume that ϑ and φ are invariant in H and that $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta) = \mathbb{F}$. If \mathbb{F} is a field of characteristic $p > 0$, assume that ϑ and φ belong to p -blocks of defect zero.

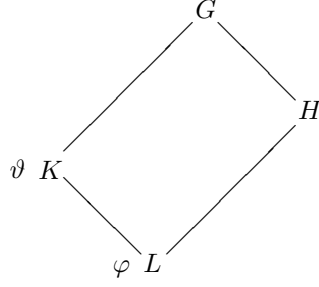


FIGURE 1. The Basic Configuration

The assumption for characteristic p assures that there are central primitive idempotents e_φ (respective e_ϑ) in $\mathbb{F}L$ (respective $\mathbb{F}K$) such that $\varphi(\mathbb{F}Le_\varphi) \neq 0$ (respective $\vartheta(\mathbb{F}Ke_\vartheta) \neq 0$) and such that $\mathbb{F}Le_\varphi$ and $\mathbb{F}Ke_\vartheta$ are central simple algebras over \mathbb{F} . In characteristic zero, this is true anyway. We set $i = e_\varphi e_\vartheta$. Observe that $i^h = i$ for all $h \in H$ and that $i^2 = e_\varphi e_\vartheta e_\varphi e_\vartheta = e_\varphi^2 e_\vartheta^2 = i$ since $e_\vartheta \in \mathbf{Z}(\mathbb{F}K)$. We write $S = (i\mathbb{F}Ki)^L = \mathbf{C}_{i\mathbb{F}Ki}(L)$. We fix this notation for the rest of this section.

3.2. Lemma. *S is a central simple \mathbb{F} -algebra, its dimension over \mathbb{F} is n^2 , and $\mathbf{C}_{i\mathbb{F}Ki}(S) = \mathbb{F}Li \cong \mathbb{F}Le_\varphi$.*

Proof. Let V be a $\mathbb{F}K$ -module affording ϑ . Then $Vi = Ve_\varphi$ is the φ -part of $V_{\mathbb{F}L}$, and thus $Vi \cong nU$ for some simple $\mathbb{F}L$ -module affording φ . (In particular, $i \neq 0$.) The isomorphisms $\mathbb{F}Ke_\vartheta \cong \text{End}_{\mathbb{F}} V \cong \mathbf{M}_{\vartheta(1)}(\mathbb{F})$ restrict to isomorphisms $i\mathbb{F}Ki \cong \text{End}_{\mathbb{F}}(Vi) \cong \mathbf{M}_{n\varphi(1)}(\mathbb{F})$. In particular, $\dim_{\mathbb{F}}(i\mathbb{F}Ki) = n^2\varphi(1)^2$.

Since $i \neq 0$, the ring homomorphism $\alpha \mapsto \alpha i = \alpha e_\vartheta$ from $\mathbb{F}Le_\varphi$ to $\mathbb{F}Li$ is not zero. As $\mathbb{F}Le_\varphi$ is simple, the map $\alpha \mapsto \alpha i$ is injective. Thus $\mathbb{F}Le_\varphi \cong \mathbb{F}Li$.

The algebras $\mathbb{F}Ke_\vartheta$ and $i\mathbb{F}Ki$ are central simple. By definition, S is just the centralizer of $\mathbb{F}Li$ in $i\mathbb{F}Ki$. By the Centralizer Theorem [9, Theorem 3.15], S is central simple, too, and the centralizer of S is again $\mathbb{F}Li$. Also $i\mathbb{F}Ki \cong S \otimes_{\mathbb{F}} \mathbb{F}Li$. From this and from $\dim_{\mathbb{F}}(i\mathbb{F}Ki) = n^2\varphi(1)^2$ it follows that $\dim_{\mathbb{F}} S = n^2$ as claimed. \square

3.3. Corollary. *$i\mathbb{F}Gi \cong S \otimes_{\mathbb{F}} C$, where $C = \mathbf{C}_{i\mathbb{F}Gi}(S)$.*

Proof. Clear by Lemma 2.1. \square

3.4. Lemma. *$\mathbb{F}Ke_\vartheta = \mathbb{F}Ki\mathbb{F}K$ and $\mathbb{F}Ge_\vartheta = \mathbb{F}Gi\mathbb{F}G$.*

Proof. The first assertion follows since $\mathbb{F}Ke_\vartheta$ is a simple ring and $i \neq 0$, and the second follows from the first. \square

By the last two results and the results from Section 2, the characters of $\mathbb{F}Ke_\vartheta$ are in bijection with the characters of C . We now work to find isomorphisms between C and $\mathbb{F}He_\varphi$. Such isomorphisms exist under additional conditions.

3.5. Lemma. *Let T be the inertia group of φ in $\mathbf{N}_G(L)$. Then T/L acts on $S = (i\mathbb{F}Ki)^L$ (by conjugation). There is a projective representation $\sigma: T/L \rightarrow S$ such that $s^g = s^{\sigma(Lg)}$ for all $s \in S$ and $g \in T$.*

Observe that by Hypothesis 3.1, we have $H \leq T$.

Proof. As ϑ is G -invariant, T acts on S and clearly L acts trivially on S . Since S is a central simple \mathbb{F} -algebra, all automorphisms are inner by the Skolem-Noether Theorem [9, Theorem 3.14]. We can thus choose $\sigma(Lg) \in S$ for every $g \in T$ such that $s^g = s^{\sigma(Lg)}$ for all $s \in S$. This determines $\sigma(Lg)$ up to multiplication with an element of $\mathbf{Z}(S) = \mathbb{F}i$. In this way we get a projective representation σ from T/L to S with the desired property. \square

We digress to prove a fact mentioned in the introduction. We write $[\vartheta]_{(G/K, \mathbb{F})}$ for the cohomology class in $H^2(G/K, \mathbb{F}^*)$ determined by ϑ . We write $[\vartheta]_{(H/L, \mathbb{F})}$ for its image in $H^2(H/L, \mathbb{F}^*)$ under the isomorphism induced by the natural isomorphism $G/K \cong H/L$.

3.6. *Remark.* Choose σ as in Lemma 3.5. Let α be the cocycle of H/L defined by

$$\sigma(x)\sigma(y) = \alpha(x, y)\sigma(xy),$$

and let $[\alpha] \in H^2(H/L, \mathbb{F}^*)$ be its cohomology class. Then

$$[\vartheta]_{(H/L, \mathbb{F})} = [\alpha][\varphi]_{(H/L, \mathbb{F})}.$$

Proof. Let us first recall the definition of $[\vartheta]_{(G/K, \mathbb{F})}$ and $[\varphi]_{(H/L, \mathbb{F})}$: For every $g \in G$ there exists $u_g \in \mathbb{F}Ke_\vartheta$ such that $a^g = a^{u_g}$ for all $a \in \mathbb{F}Ke_\vartheta$. This can be done such that $u_{gk} = u_gk$ and $u_{kg} = ku_g$ for all $k \in K$ and $g \in G$. Define $f \in Z^2(G, \mathbb{F}^*)$ by $u_x u_y = f(x, y)u_{xy}$. Then f is constant on cosets of K and thus may be viewed as an element of $Z^2(G/K, \mathbb{F}^*)$. Its cohomology class is, by definition, $[\vartheta]_{(G/K, \mathbb{F})}$. (A more usual approach is probably this [15, Theorem 11.2]: Suppose ϑ is afforded by a representation $D: K \rightarrow \mathbf{M}_{\vartheta(1)}(\mathbb{F})$. (This may not be the case!) Extended D to a projective representation of G . The corresponding factor set f defines $[\vartheta]_{(G/K, \mathbb{F})}$. Note that if u_g as above are defined, we may extend D by setting $\widehat{D}(g) = D(u_g)$.)

Similarly, choose $v_h \in \mathbb{F}Le_\varphi$ such that $b^h = b^{v_h}$ for all $b \in \mathbb{F}Le_\varphi$.

Since $i\mathbb{F}Ki \cong S \otimes_{\mathbb{F}} \mathbb{F}Le_\varphi$, we get that for $s \in S$ and $b \in \mathbb{F}Le_\varphi$, we have

$$(sb)^{\sigma(h)v_h} = s^{\sigma(h)}b^{v_h} = s^h b^h = (sb)^h = (sb)^{u_h}.$$

Since $i\mathbb{F}Ki$ is central simple, it follows that $iu_h = \lambda_h \sigma(h)v_h$ with $\lambda_h \in \mathbb{F}$. The proof follows. \square

3.7. **Definition.** In the situation of Hypothesis 3.1, we say that

$$\sigma: H/L \rightarrow S = (i\mathbb{F}Ki)^L$$

is a magic representation (for $G, H, K, L, \vartheta, \varphi$), if

- (a) $\sigma(Lh_1 h_2) = \sigma(Lh_1)\sigma(Lh_2)$ for all $h_1, h_2 \in H$ and
- (b) $s^h = s^{\sigma(Lh)}$ for all $s \in S$ and $h \in H$.

The character of a magic representation, that is the function $\psi: H/L \rightarrow \mathbb{F}$ defined by $\psi(Lh) = \text{tr}_{S/\mathbb{F}}(\sigma(Lh))$, is called a magic character.

3.8. **Theorem.** Assume Hypothesis 3.1 and let $\sigma: H/L \rightarrow S$ be a magic representation. Then the linear map

$$\kappa: \mathbb{F}H \rightarrow C = \mathbf{C}_{i\mathbb{F}Gi}(S), \quad \text{defined by } h \mapsto h\sigma(Lh)^{-1} \text{ for } h \in H,$$

is an algebra-homomorphism and induces an isomorphism $\mathbb{F}He_\varphi \cong C$.

Proof. For $h \in H$ let $c_h = h\sigma(Lh)^{-1} = \sigma(Lh)^{-1}h$. (The inverse $\sigma(Lh)^{-1}$ is the inverse in S , so $\sigma(Lh)\sigma(Lh)^{-1} = i$.) Clearly $c_h \in C$. Note that

$$\begin{aligned} c_g c_h &= g\sigma(Lg)^{-1}h\sigma(Lh)^{-1} = gh(\sigma(Lg)^{-1})^{\sigma(Lh)}\sigma(Lh)^{-1} \\ &= gh\sigma(Lh)^{-1}\sigma(Lg)^{-1} = gh\sigma(Lgh)^{-1} = c_{gh}. \end{aligned}$$

Thus extending the map $h \mapsto c_h$ linearly to $\mathbb{F}H$ defines an algebra homomorphism $\kappa: \mathbb{F}H \rightarrow C$. For $l \in L$ we have $l \mapsto l\sigma(L)^{-1} = li = l \cdot 1_C$. Thus κ restricted to $\mathbb{F}L$ is just multiplication with i , so that $e_\varphi \kappa = e_\varphi i = i = 1_C$, and any other central idempotent of $\mathbb{F}L$ maps to zero. Therefore

$$(\mathbb{F}L)\kappa = (\mathbb{F}Le_\varphi)\kappa = \mathbb{F}Li.$$

For any $h \in H$ we have $(\mathbb{F}Le_\varphi h)\kappa = \mathbb{F}Lic_h$. Let T be a transversal for the cosets of L in H . As $\mathbb{F}He_\varphi = \bigoplus_{t \in T} \mathbb{F}Le_\varphi t$, the proof will be finished if we show that

$C = \bigoplus_{t \in T} \mathbb{F}Li_t$. But this follows from standard arguments from the theory of group graded algebras: The decomposition $\mathbb{F}G = \bigoplus_{t \in T} \mathbb{F}Kt$ yields the decomposition $i\mathbb{F}Gi = \bigoplus_{t \in T} i\mathbb{F}Kti$. For $Kg \in G/K$ set $C_{Kg} = C \cap \mathbb{F}Kg = \mathbf{C}_{i\mathbb{F}Kg i}(S)$. Since $S \subseteq i\mathbb{F}Ki$, the centralizer C of S inherits the grading of $i\mathbb{F}Gi$:

$$C = \bigoplus_{Kg \in G/K} C_{Kg} = \bigoplus_{t \in T} C_{Kt}.$$

As $c_t \in C \cap i\mathbb{F}Kti = C \cap i\mathbb{F}Kti = C_{Kt}$ and c_t is a unit of C , we conclude that

$$C_{Kt} = C_{Kt}c_t^{-1}c_t \subseteq C_Kc_t \subseteq C_{Kt},$$

so equality holds throughout. As $C_K = \mathbf{C}_{i\mathbb{F}Ki}(S) = \mathbb{F}Li$ by Lemma 3.2, the proof follows. \square

Remark. If the projective representation of Lemma 3.5 is not equivalent to an ordinary representation, we still get some result. Let α be a factor set associated with σ , that is, $\sigma(x)\sigma(y) = \alpha(x, y)\sigma(xy)$ for $x, y \in H/L$. We may view α as a factor set of H which is constant on cosets of L . Let $\mathbb{F}^{\alpha^{-1}}[H]$ denote the twisted group algebra with respect to α^{-1} , that is, $\mathbb{F}^{\alpha^{-1}}[H]$ has a basis $\{b_h \mid h \in H\}$ where $b_h b_g = \alpha(h, g)^{-1} b_{hg}$. The ordinary group algebra $\mathbb{F}L$ can be embedded in the twisted group algebra $\mathbb{F}^{\alpha^{-1}}[H]$ in an obvious way, and thus it is meaningful to view e_φ as an idempotent in $\mathbb{F}^{\alpha^{-1}}[H]$. With this notation, nearly the same proof as above shows that $C \cong \mathbb{F}^{\alpha^{-1}}[H]e_\varphi$.

The following is an immediate consequence:

3.9. Corollary. *Assume Hypothesis 3.1 and that there is a magic representation for this configuration. Then $i\mathbb{F}Gi \cong S \otimes_{\mathbb{F}} \mathbb{F}He_\varphi$ and if $S \cong \mathbf{M}_n(\mathbb{F})$, then $\mathbb{F}Ge_\vartheta$ and $\mathbb{F}He_\varphi$ are Morita equivalent.*

Proof. The first assertion follows by combining Theorem 3.8 with Corollary 3.3. If $S \cong \mathbf{M}_n(\mathbb{F})$, then $\mathbb{F}He_\varphi$ and $i\mathbb{F}Gi \cong \mathbf{M}_n(\mathbb{F}He_\varphi)$ are Morita-equivalent, and $\mathbb{F}Ge_\vartheta$ and $i\mathbb{F}Gi$ are Morita-equivalent, since i is a full idempotent in $\mathbb{F}Ge_\vartheta$ by Lemma 3.4. \square

4. CHARACTER CORRESPONDENCES

If $\mathbb{F} \leq \mathbb{C}$, then every magic representation yields a character correspondence between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$. We need some additional general notation to state the properties of this correspondence.

4.1. Notation.

- (a) For any central simple \mathbb{F} -algebra S , let $[S]$ denote the algebra equivalence class of S in the Brauer group of \mathbb{F} .
- (b) For $\chi \in \text{Irr } G$ and $\mathbb{F} \leq \mathbb{C}$, set $[\chi]_{\mathbb{F}} = [\mathbb{F}(\chi)Ge_\chi]$. This is an element of the Brauer group of $\mathbb{F}(\chi)$.

4.2. Notation. For $\gamma: G \rightarrow \mathbb{C}$ a class function and ϑ an irreducible character of some subgroup, let γ_ϑ be the class function defined by

$$\gamma_\vartheta = \sum_{\chi \in \text{Irr}(G \mid \vartheta)} (\gamma, \chi)_G \chi.$$

This is the part of γ that lies above ϑ . We will need the case where ϑ is an irreducible character of some normal subgroup and invariant in G . Note that then $\gamma_\vartheta(g) = \gamma(ge_\vartheta)$.

4.3. Theorem. *Assume Hypothesis 3.1 with $\mathbb{F} \leq \mathbb{C}$. Let $\sigma: H/L \rightarrow S^*$ be a magic representation. Pick $s_0 \in S$ with $\text{tr}_{S/\mathbb{F}}(s_0) = 1$. Let $U \leq G$ with $K \leq U$. For $\chi \in \mathbb{C}[\text{Irr}(U \mid \vartheta)]$, define*

$$(1) \quad \chi^\iota(h) = \chi(s_0 \sigma(Lh)^{-1} h).$$

Then $\iota = \iota(\sigma): \mathbb{C}[\text{Irr}(U \mid \vartheta)] \rightarrow \mathbb{C}[\text{Irr}(U \cap H \mid \varphi)]$ is an isomorphism independent of the choice of s_0 . Let ψ be the character of σ . The correspondence ι has the following properties:

- (a) $\chi \in \text{Irr}(U \mid \vartheta)$ if and only if $\chi^\iota \in \text{Irr}(U \cap H \mid \varphi)$.
- (b) $(\chi_1^\iota, \chi_2^\iota)_{U \cap H} = (\chi_1, \chi_2)_U$ for $\chi_1, \chi_2 \in \mathbb{C}[\text{Irr}(U \mid \vartheta)]$.
- (c) $\chi(1)/\vartheta(1) = \chi^\iota(1)/\varphi(1)$.
- (d) $(\chi_{U_1})^\iota = (\chi^\iota)_{U_1 \cap H}$ for $K \leq U_1 \leq U_2 \leq G$ and $\chi \in \mathbb{C}[\text{Irr}(U_2 \mid \vartheta)]$.
- (e) $(\tau^{U_2})^\iota = (\tau^\iota)^{U_2 \cap H}$ for $K \leq U_1 \leq U_2 \leq G$ and $\tau \in \mathbb{C}[\text{Irr}(U_1 \mid \vartheta)]$.
- (f) $(\chi^\iota)^h = (\chi^h)^\iota$ for $h \in H$.
- (g) $(\beta\chi)^\iota = \beta\chi^\iota$ for all $\beta \in \mathbb{C}[\text{Irr}(U/K)]$.
- (h) If α is a field automorphism fixing \mathbb{F} , then $(\chi^\alpha)^\iota = (\chi^\iota)^\alpha$; and $\mathbb{F}(\chi) = \mathbb{F}(\chi^\iota)$.
- (i) $[\chi]_{\mathbb{F}} = [S \otimes_{\mathbb{F}} \mathbb{F}(\chi)][\chi^\iota]_{\mathbb{F}}$ for $\chi \in \text{Irr}(U \mid \vartheta)$.
- (j) $(\chi_{U \cap H})_\varphi = \psi\chi^\iota$ for $\chi \in \mathbb{C}[\text{Irr}(U \mid \vartheta)]$.
- (k) $(\xi^U)_\vartheta = \overline{\psi}\xi^{\iota^{-1}}$ for $\xi \in \mathbb{C}[\text{Irr}(U \cap H \mid \varphi)]$. (Here we view ψ as character on G/K , via the canonical isomorphism $G/K \cong H/L$.)

Proof. Note that

$$\mathbb{C}[\text{Irr}(G \mid \vartheta)] \cong \text{ZF}(\text{CG}e_\vartheta, \mathbb{C}) \quad \text{and} \quad \mathbb{C}[\text{Irr}(H \mid \varphi)] \cong \text{ZF}(\text{CH}e_\varphi, \mathbb{C})$$

naturally. We work first over \mathbb{C} . Set $S_{\mathbb{C}} = (i\text{CK}i)^L$ and $C_{\mathbb{C}} = \mathbf{C}_{i\text{CG}i}(S_{\mathbb{C}})$.

By Lemma 3.4 we have $\text{CG}e_\vartheta = \text{CG}i\text{CG}$. Lemma 2.7 yields that restriction defines an isomorphism $\text{ZF}(\text{CG}e_\vartheta, \mathbb{C}) \rightarrow \text{ZF}(i\text{CG}i, \mathbb{C})$.

Since $S_{\mathbb{C}} = (i\text{CK}i)^L \cong \mathbf{M}_n(\mathbb{C})$ by Lemma 3.2, Lemma 2.4 yields an isomorphism $\varepsilon: \text{ZF}(i\text{CG}i, \mathbb{C}) \rightarrow \text{ZF}(C_{\mathbb{C}}, \mathbb{C})$.

Finally, the isomorphism $\kappa: \text{CH}e_\varphi \rightarrow C_{\mathbb{C}}$ of Theorem 3.8 (applied with $\mathbb{F} = \mathbb{C}$) yields an isomorphism $\kappa^*: \text{ZF}(C_{\mathbb{C}}, \mathbb{C}) \rightarrow \text{ZF}(\text{CH}e_\varphi, \mathbb{C})$.

We claim that ι is the composition of these three isomorphisms:

$$\text{ZF}(\text{CG}e_\vartheta, \mathbb{C}) \xrightarrow[(2.7)]{\text{Res}} \text{ZF}(i\text{CG}i, \mathbb{C}) \xrightarrow[(2.4)]{\varepsilon} \text{ZF}(C_{\mathbb{C}}, \mathbb{C}) \xrightarrow[(3.8)]{\kappa^*} \text{ZF}(\text{CH}e_\varphi, \mathbb{C})$$

To see this, let $h \in H$ and $\chi \in \mathbb{C}[\text{Irr}(G \mid \vartheta)]$. Then

$$\chi^{\text{Res} \varepsilon \kappa^*}(h) = \chi^\varepsilon(\sigma(Lh)^{-1} h) = \chi(s_0 \sigma(Lh)^{-1} h).$$

Moreover, Lemma 2.4 yields that ε , and thus $\iota = \text{Res} \cdot \varepsilon \kappa^*$, is independent of the choice of s_0 .

It follows from Lemma 2.7 and Lemma 2.5 that the first two isomorphisms send characters of (irreducible) modules to characters of (irreducible) modules. This is true for κ^* , too, since κ is an isomorphism. Thus $\iota: \text{ZF}(\text{CG}e_\vartheta, \mathbb{C}) \rightarrow \text{ZF}(\text{CH}e_\varphi, \mathbb{C})$ is an isomorphism that sends irreducible characters to irreducible characters. It follows that ι respects the inner product on the space of class functions on G respective H .

Of course our reasoning so far applies to any subgroup U with $K \leq U \leq G$ instead of G , and to $V = H \cap U$ instead of H . Thus we get an isometry $\iota: \text{ZF}(\text{CU}e_\vartheta, \mathbb{C}) \rightarrow \text{ZF}(\text{CV}e_\varphi, \mathbb{C})$ for every such subgroup U . We use the same letter ι to denote all these isometries and their union. We have now established that ι is well defined and bijective, as well as Properties (a) and (b) of ι .

To show that (c) holds, we choose $s_0 = (1/n)i = (1/n)e_{\vartheta}e_{\varphi}$. Remember that $n = (\vartheta_L, \varphi)_L$, so that $\vartheta(e_{\varphi}) = n\varphi(1)$. Thus

$$\frac{\chi^{\iota}(1)}{\varphi(1)} = \frac{\chi((1/n)e_{\vartheta}e_{\varphi})}{\varphi(1)} = \frac{\chi(e_{\varphi})}{n\varphi(1)} = \frac{(\chi_K, \vartheta)\vartheta(e_{\varphi})}{n\varphi(1)} = (\chi_K, \vartheta) = \frac{\chi(1)}{\vartheta(1)}.$$

Properties (d)–(h) are consequences of a general theorem about graded Morita equivalences [22, Theorem 3.4], which applies here. It is, however, not difficult to prove them directly. We do this for (g) and (h).

Let $\beta \in \mathbb{C}[\text{Irr}(G/K)]$. Then

$$(\beta\chi)^{\iota}(h) = (\beta\chi)(s_0\sigma(h)^{-1}h).$$

Writing $s_0\sigma(h)^{-1} = \sum_{k \in K} \lambda_k k$ with $\lambda_k \in \mathbb{F}$, we get

$$\begin{aligned} (\beta\chi)^{\iota}(h) &= \sum_{k \in K} \lambda_k \beta(kh) \chi(kh) = \beta(h) \sum_{k \in K} \lambda_k \chi(kh) \\ &= \beta(h) \chi(s_0\sigma(h)^{-1}h) = \beta(h) \chi^{\iota}(h). \end{aligned}$$

This proves (g).

To prove (h), write $s_0\sigma(h)^{-1} = \sum_{k \in K} \lambda_k k$ as above. Since $\lambda_k \in \mathbb{F}$, it follows

$$\begin{aligned} (\chi^{\alpha})^{\iota}(h) &= \chi^{\alpha}(s_0\sigma(Lh)^{-1}h) = \sum_{k \in K} \lambda_k \chi^{\alpha}(kh) \\ &= \sum_{k \in K} \lambda_k \chi(kh)^{\alpha} = \left(\sum_{k \in K} \lambda_k \chi(kh) \right)^{\alpha} \\ &= (\chi^{\iota}(h))^{\alpha}. \end{aligned}$$

Thus $(\chi^{\alpha})^{\iota} = (\chi^{\iota})^{\alpha}$, and $\mathbb{F}(\chi) = \mathbb{F}(\chi^{\iota})$ follows from this.

Now let $\chi \in \text{Irr}(G \mid \vartheta)$. Then $[\chi]_{\mathbb{F}}$ is the equivalence class of $\mathbb{F}(\chi)Ge_{\chi}$ in the Brauer group of $\mathbb{F}(\chi)$. By (h) we may assume that $\mathbb{F} = \mathbb{F}(\chi) = \mathbb{F}(\chi^{\iota})$. The isomorphism of Corollary 3.9 sends $S \otimes_{\mathbb{F}} \mathbb{F}He_{\chi^{\iota}}$ onto $i\mathbb{F}Gie_{\chi} = i\mathbb{F}Ge_{\chi}i$. But this central simple algebra is in the equivalence class of $\mathbb{F}Ge_{\chi}$, since ie_{χ} is an idempotent. This proves (i).

Property (j): For $\chi \in \mathbb{C}[\text{Irr}(G \mid \vartheta)]$ we have

$$(\chi_H)_{\varphi}(h) = \chi(he_{\varphi}) = \chi(he_{\varphi}e_{\vartheta}) = \chi(hi).$$

Now note that $hi = ih = \sigma(Lh)\sigma(Lh)^{-1}h$ for $h \in H$. By Lemma 2.4 and the definition of ι , we have

$$\chi(hi) = \text{tr}_{S/\mathbb{F}}(\sigma(Lh))\chi(s_0\sigma(Lh)^{-1}h) = \psi(h)\chi^{\iota}(h)$$

as claimed in (j).

To prove (k), it suffices to show that $(\overline{\psi}\xi^{\iota^{-1}}, \chi)_G = (\xi^G, \chi)_G$ for all $\chi \in \text{Irr}(G \mid \vartheta)$. Using what we have already proved, we get

$$\begin{aligned} (\overline{\psi}\xi^{\iota^{-1}}, \chi)_G &= (\xi^{\iota^{-1}}, \psi\chi)_G \\ &= (\xi, (\psi\chi)^{\iota})_H \quad (\text{as } \iota \text{ is an isometry}) \\ &= (\xi, \psi\chi^{\iota})_H \quad (\text{by (g).}) \\ &= (\xi, (\chi_H)_{\varphi})_H \quad (\text{by (j).}) \\ &= (\xi, \chi_H)_H \quad (\text{as } \xi \in \mathbb{C}[\text{Irr}(H \mid \varphi)]) \\ &= (\xi^G, \chi)_G \end{aligned}$$

as was to be shown. The proof is complete. \square

The bijection of the theorem depends on the magic representation σ . If such a representation exists, it is unique up to multiplication with a linear character of H/L (with values in \mathbb{F}). Different choices of σ give bijections which differ by multiplication with a linear character of H/L . Note that if $\psi(h) \neq 0$ for all $h \in H$, then χ^ι is determined by the equation $(\chi_H)_\varphi = \psi\chi^\iota$. Otherwise one needs the representation σ to compute χ^ι from Equation (1).

To formulate the next result correctly, we need the *reduced norm* of a central simple algebra S over \mathbb{F} , which we denote by $\text{nr}_{S/\mathbb{F}}$ or simply nr , if S and \mathbb{F} are clear from context. Remember that it is defined as follows: First, choose a splitting field $\mathbb{E} \geq \mathbb{F}$ of S and an isomorphism $\varepsilon: S \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbf{M}_n(\mathbb{E})$. Then for $s \in S$ define $\text{nr}_{S/\mathbb{F}}(s) = \det(\varepsilon(s \otimes 1))$, where \det denotes the determinant of the matrix ring $\mathbf{M}_n(\mathbb{E})$. It can be shown that $\text{nr}(s)$ is independent of the particular isomorphism ε and of the choice of \mathbb{E} , and that $\text{nr}(s) \in \mathbb{F}$ [25, § 9a]. If $\sigma: H/L \rightarrow S^*$ is a magic representation, then $x \mapsto \text{nr}(\sigma(x))$ defines a linear character, which we denote simply by $\det \sigma$.

4.4. Remark. Let π be the set of prime divisors of n . If there is any magic representation, then there is a magic representation σ such that $\det \sigma$ has order a π -number.

Proof. Suppose $\sigma: H/L \rightarrow S$ is given. Let $\lambda = \det \sigma$, a linear character of H/L . Let b be the π' -part of $\mathbf{o}(\lambda)$. As $n = \dim \sigma$ is π , there is $r \in \mathbb{Z}$ with $rn + 1 \equiv 0 \pmod{b}$. Then $\det(\lambda^r \sigma) = \lambda^{rn+1}$ has π -order. \square

Let $\alpha \in \text{Aut } \mathbb{F}$ be a field automorphism. Then α extends naturally to an automorphism of the group algebra $\mathbb{F}G$, which we denote also by α . Remember that $\text{Aut } \mathbb{F}$ acts on the set of class functions $\chi: G \rightarrow \mathbb{F}$ by $\chi^\alpha(g) = \chi(g)^\alpha$ ($g \in G$). As usual, a class function extends linearly to a function $\mathbb{F}G \rightarrow \mathbb{F}$. Note that then $\chi^\alpha(c^\alpha) = \chi(c)^\alpha$ for $c = \sum_g c_g g \in \mathbb{F}G$ arbitrary. This will be used in the proof of the next proposition.

4.5. Proposition. Let $\mathcal{B} = (G, H, K, L, \vartheta, \varphi)$ be a configuration such that Hypothesis 3.1 holds over the field \mathbb{F} , and let $\alpha \in \text{Aut } \mathbb{F}$. If $\sigma: H/L \rightarrow S$ is a magic representation for \mathcal{B} with magic character ψ , then

$$\sigma^\alpha: H/L \rightarrow S \quad \text{defined by} \quad \sigma^\alpha(h) = \sigma(h)^\alpha$$

is a magic representation for $\mathcal{B}^\alpha = (G, H, K, L, \vartheta^\alpha, \varphi^\alpha)$ with magic character ψ^α . Let $\iota(\sigma)$ and $\iota(\sigma^\alpha)$ be the associated character correspondences. Then

$$(\chi^\alpha)^{\iota(\sigma^\alpha)} = \left(\chi^{\iota(\sigma)} \right)^\alpha \quad \text{for} \quad \chi \in \text{Irr}(G \mid \vartheta).$$

Proof. Note that $e_\vartheta^\alpha = e_{\vartheta^\alpha}$ for the central primitive idempotent belonging to ϑ . Thus $\sigma^\alpha: H/L \rightarrow S^\alpha = (i^\alpha \mathbb{F}^\alpha K i^\alpha)^L$ with $i^\alpha = e_{\vartheta^\alpha} e_{\varphi^\alpha}$ is a magic representation for the configuration \mathcal{B}^α .

The isomorphism α maps the reduced trace of S to the reduced trace of S^α , by uniqueness of the reduced trace, and so ψ^α is the character of σ^α .

Let $\chi \in \text{Irr}(G \mid \vartheta)$ and pick $s_0 \in S$ with reduced trace 1. Thus $\text{tr}_{S^\alpha/\mathbb{F}}(s_0^\alpha) = 1$. Therefore

$$\begin{aligned} (\chi^\alpha)^{\iota(\sigma^\alpha)}(h) &= \chi^\alpha(s_0^\alpha \sigma^\alpha(Lh)^{-1} h) = \chi^\alpha((s_0 \sigma(Lh)^{-1} h)^\alpha) \\ &= \chi(s_0 \sigma(Lh)^{-1} h)^\alpha = \chi^{\iota(\sigma)}(h)^\alpha = \left(\chi^{\iota(\sigma)} \right)^\alpha(h), \end{aligned}$$

as was to be shown. \square

4.6. Proposition. Assume Hypothesis 3.1 and let $C \leq \mathbf{C}_H(S)$ with $L \leq C$.

- (a) For every $\chi \in \text{Irr}(KC \mid \vartheta)$, there is a unique $\xi \in \text{Irr}(C \mid \varphi)$ such that $(\chi_C, \xi)_C > 0$. This defines a bijection between $\text{Irr}(KC \mid \vartheta)$ and $\text{Irr}(C \mid \varphi)$, which has all the properties of Theorem 4.3 with $\psi = n1_{C/L}$. This bijection is invariant under all automorphisms of G fixing K , L , ϑ and φ .
- (b) Assume that $C \trianglelefteq H$, that $\xi \in \text{Irr}(C \mid \varphi)$ is H -stable and χ corresponds to ξ . Then $T = (e_\xi \mathbb{F} K C e_\chi e_\xi)^C$ and $S = (e_\varphi \mathbb{F} K e_\vartheta e_\varphi)^L$ are isomorphic as H/C -algebras.

Note that if $n = 1$, then $\mathbf{C}_H(S) = H$. This case of Part (a) is known and can be proved just using the orthogonality relations of ordinary character theory [14, Lemma 4.1].

Proof. Theorem 4.3 applies to the configuration $(KC, C, K, L, \vartheta, \varphi)$ with $\sigma: C/L \rightarrow S$, $\sigma(c) = 1_S = i$ for all $c \in C$. Observe that then $\psi = n1_C$. From Property (j) in Theorem 4.3 it now follows that the restriction χ_C of every $\chi \in \text{Irr}(KC \mid \vartheta)$ has a unique constituent in $\text{Irr}(C \mid \varphi)$, which occurs with multiplicity n , as claimed. Conversely, for $\xi \in \text{Irr}(C \mid \varphi)$, the induced character ξ^G has a unique constituent lying in $\text{Irr}(KC \mid \vartheta)$, by Property (k). The desired bijection is thus just the correspondence ι of Theorem 4.3. Part (a) follows.

Let $j = e_\chi e_\xi$, where we assume that ξ and χ are H -invariant. Thus $T = (j \mathbb{F} K C j)^C$. The idempotent j centralizes S , as e_χ is in the center of $\mathbb{F} K C$, and $e_\xi \in \mathbb{F} C$ with $C \leq \mathbf{C}_H(S)$. It follows that for every $s \in S$, we have $sj = js = jsj \in T$. As $e_\chi e_\vartheta = e_\chi$ and $e_\xi e_\varphi = e_\xi$, it follows that $ij = j$. The map $s \mapsto sj$ is thus an algebra homomorphism from S into T . Since S is simple and $\dim_{\mathbb{F}} S = n = \dim_{\mathbb{F}} T$, the map is an isomorphism. It is compatible with the action of H as j is H -stable. \square

4.7. Remark. Part (a) says that $|\text{Irr}(KC \mid \vartheta) \cap \text{Irr}(KC \mid \xi)| = 1$ for all $\xi \in \text{Irr}(C \mid \varphi)$. This contains the well known description of the characters of central products. Namely, let G be the central product of K and C , set $L = K \cap C$ (so $L \leq \mathbf{Z}(G)$), and suppose $\varphi \in \text{Irr } L$. Then Part (a) applies with $H = C$ for every $\vartheta \in \text{Irr}(K \mid \varphi)$, since C centralizes K . We get a bijection between $\text{Irr}(KC \mid \varphi)$ and $\text{Irr}(K \mid \varphi) \times \text{Irr}(C \mid \varphi)$.

Part (b) has an interesting consequence, a “going down” result:

4.8. Corollary. *In the situation of Part (b) of Proposition 4.6, assume that there exists a magic representation for the configuration G, H, KC, C, χ, ξ . Suppose $L \leq D \leq C$, let $\tilde{\chi} \in \text{Irr}(KD \mid \vartheta)$ and $\tilde{\xi} \in \text{Irr}(LD \mid \varphi)$ and assume $(\tilde{\chi}, \tilde{\xi})_D > 0$. Suppose $D \trianglelefteq H$ and $\tilde{\xi}$ is invariant in H . Then there is a magic representation σ for the configuration $(G, H, KD, D, \tilde{\chi}, \tilde{\xi})$ with $C/D \leq \ker \sigma$.*

Proof. Clear since

$$(e_{\tilde{\xi}} \mathbb{F} K D e_{\tilde{\chi}} e_{\tilde{\xi}})^D \cong S \cong (e_{\xi} \mathbb{F} K C e_{\chi} e_{\xi})^C$$

as H/C -algebras for all such configurations. \square

In terms of cohomology classes and in view of Remark 3.6, this means that if $[\chi]_{(H/C, \mathbb{F})} = [\xi]_{(H/C, \mathbb{F})}$, then $[\tilde{\chi}]_{(H/D, \mathbb{F})} = [\tilde{\xi}]_{(H/D, \mathbb{F})}$. This is related to some results obtained in [16].

Note that if the configuration $(G, H, K, L, \vartheta, \varphi)$ admits a magic representation σ such that $C/L \leq \ker \sigma$, then σ may be viewed as a magic representation for the configuration (G, H, KC, C, χ, ξ) . It may be possible, however, that there are magic representations for the configuration of ϑ and φ , but no magic representation whose kernel contains C/L .

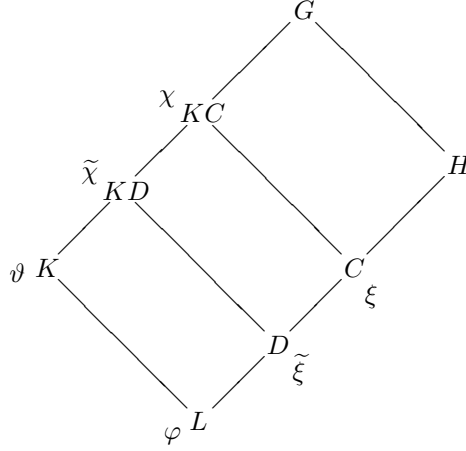


FIGURE 2. Corollary 4.8

5. SEMI-INVARIANT CHARACTERS

Throughout this section, let \mathbb{F} be a field with algebraic closure $\overline{\mathbb{F}}$. Let $K \leq G$ and $\vartheta \in \text{Irr}_{\overline{\mathbb{F}}}(K)$. If $p = \text{char } \mathbb{F} > 0$, then we assume that ϑ has p -defect zero. There is a unique central primitive idempotent e_{ϑ} of $\overline{\mathbb{F}}K$, such that ϑ does not vanish on $\overline{\mathbb{F}}Ke_{\vartheta}$. The assumption assures that $\overline{\mathbb{F}}Ke_{\vartheta} \cong \mathbf{M}_d(\overline{\mathbb{F}})$ where d is the dimension of a module affording ϑ .

If α is an automorphism of a field \mathbb{E} , then we denote also by α its natural extension to the group algebra $\mathbb{E}G$, where α centralizes G . We need the following well known fact.

5.1. Lemma.

$$e = T_{\mathbb{F}}^{\mathbb{F}(\vartheta)}(e_{\vartheta}) := \sum_{\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} (e_{\vartheta})^{\alpha}$$

is the unique central primitive idempotent of $\mathbb{F}K$ for which $\vartheta(\mathbb{F}Ke) \neq 0$.

Proof. Let $\Gamma = \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. Note that $e_{\vartheta}^{\alpha} = (e_{\vartheta})^{\alpha}$ for $\alpha \in \Gamma$. Obviously, only the identity of Γ fixes ϑ or e_{ϑ} . Thus for $\alpha \neq \beta \in \Gamma$, we have $(e_{\vartheta})^{\alpha}(e_{\vartheta})^{\beta} = 0$, so e is an idempotent.

It is clear that $e \in \mathbf{Z}(\mathbb{F}K)$ and that ϑ does not vanish on $\mathbb{F}Ke$. (Otherwise ϑ would vanish on $\overline{\mathbb{F}}Ke_{\vartheta}$.) We claim that e is a primitive idempotent in $\mathbf{Z}(\mathbb{F}K)$. Indeed, if $0 \neq f \in \mathbf{Z}(\mathbb{F}K)$ is an idempotent with $fe = f$, then $e_{\vartheta}^{\alpha}f = e_{\vartheta}^{\alpha}$ for some $\alpha \in \Gamma$. Then for any $\sigma \in \Gamma$, we have $(e_{\vartheta})^{\alpha\sigma}f = (e_{\vartheta})^{\alpha\sigma}f\sigma = (e_{\vartheta}^{\alpha}f)^{\sigma} = (e_{\vartheta}^{\alpha})^{\sigma}$, and thus $f = e$. \square

5.2. Notation. We write $e_{(\vartheta, \mathbb{F})}$ for the idempotent of Lemma 5.1. In particular, if $\mathbb{F} = \mathbb{F}(\vartheta)$, then $e_{(\vartheta, \mathbb{F})} = e_{\vartheta}$.

5.3. Lemma.

$$\mathbb{F}G_{\vartheta}e_{(\vartheta, \mathbb{F})} \ni a \mapsto ae_{\vartheta} \in \mathbb{F}(\vartheta)G_{\vartheta}e_{\vartheta}$$

is an isomorphism of \mathbb{F} -algebras.

Proof. Since $e_{\vartheta} \in \mathbf{Z}(\mathbb{F}(\vartheta)G_{\vartheta})$, the map is multiplicative.

The inverse is given by the field trace $T = T_{\mathbb{F}}^{\mathbb{F}(\vartheta)}$, extended from $\mathbb{F}(\vartheta)$ to $\mathbb{F}(\vartheta)G$ and then restricted to $\mathbb{F}(\vartheta)G_{\vartheta}e_{\vartheta}$: If $b \in \mathbb{F}(\vartheta)G_{\vartheta}e_{\vartheta}$, then $b^{\alpha} \in \mathbb{F}(\vartheta)G_{\vartheta}e_{\vartheta}^{\alpha}$ for $\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. If $\alpha \neq 1$, then $b^{\alpha}e_{\vartheta} = 0$. Thus $T(b)e_{\vartheta} = be_{\vartheta} = b$. Conversely, for $a \in \mathbb{F}G_{\vartheta}e_{(\vartheta, \mathbb{F})}$ we have $T(ae_{\vartheta}) = aT(e_{\vartheta}) = ae_{(\vartheta, \mathbb{F})} = a$, since $T(e_{\vartheta}) = e_{(\vartheta, \mathbb{F})}$. \square

As a consequence, $\mathbf{Z}(\mathbb{F}K e_{(\vartheta, \mathbb{F})}) \cong \mathbb{F}(\vartheta)$. This is of course well known. An isomorphism is given by the central character ω_ϑ .

The following notation will be convenient: Let $K \trianglelefteq G$ and $e \in \mathbf{Z}(\mathbb{F}K)$. Let $G_e = \{g \in G \mid e^g = e\}$ and write e^G for the idempotent defined by $e^G := T_{G_e}^G(e) = \sum_{g \in [G:G_e]} e^g$.

The following result is also well known:

5.4. Proposition. *Set $e = e_{(\vartheta, \mathbb{F})}$ and $f = e^G$ and let $T = G_e$ be the inertia group of e . Then $\mathbb{F}Gf \cong \mathbf{M}_{|G:T|}(\mathbb{F}Te)$. Induction defines a bijection between $\text{Irr}(T \mid \vartheta)$ and $\text{Irr}(G \mid \vartheta)$ that preserves field of values and Schur indices over \mathbb{F} . More precisely, $\mathbb{F}(\tau^G) = \mathbb{F}(\tau)$ and $[\tau^G]_{\mathbb{F}} = [\tau]_{\mathbb{F}}$ for $\tau \in \text{Irr}(T \mid \vartheta)$.*

(Here $[\tau]_{\mathbb{F}}$ is the equivalence class in the Brauer group associated with τ , see 4.1.)

Proof. Let $G = \bigcup_{u \in R} Tu$. Since $e^g \neq e$ for $g \in G \setminus T$, it follows $e^g e = 0$ and thus $e g e = 0$. From this it follows easily that $\{E_{u,v} = u^{-1} e v \mid u, v \in R\}$ is a full set of matrix units in $\mathbb{F}Gf$ and that $e \mathbb{F}G e = \mathbb{F}T e$. It is then routine to verify that

$$\mathbb{F}Gf \ni a \mapsto (e u a v^{-1} e)_{u,v \in R} \in \mathbf{M}_{|G:T|}(\mathbb{F}T e)$$

and

$$\mathbf{M}_{|G:T|}(\mathbb{F}T e) \ni (b_{u,v})_{u,v} \mapsto \sum_{u,v \in R} u^{-1} b_{u,v} v \in \mathbb{F}G e \mathbb{F}G = \mathbb{F}G f$$

are inverse isomorphisms.

This isomorphism yields a bijection between isomorphism classes of $\mathbb{F}Gf$ -modules and $\mathbb{F}T e$ -modules. Let V be an $\mathbb{F}T e$ -module. Then $V^{|G:T|}$ is a module over $\mathbf{M}_{|G:T|}(\mathbb{F}T e) \cong \mathbb{F}Gf$, and $V^{|G:T|}$ as $\mathbb{F}Gf$ -module is isomorphic to $V \otimes_{\mathbb{F}T} \mathbb{F}G$ via the map $(v_u)_{u \in R} \mapsto \sum_{u \in R} v_u \otimes u$. Thus induction yields a bijection between isomorphism classes of $\mathbb{F}T e$ -modules and $\mathbb{F}Gf$ -modules. (Alternatively, one can check directly that $(V \otimes_{\mathbb{F}T} \mathbb{F}G)e \cong V$ as $\mathbb{F}T$ -modules, and that, if W is an $\mathbb{F}Gf$ -module, then $W e \otimes_{\mathbb{F}T} \mathbb{F}G \cong W$ as $\mathbb{F}G$ -modules.)

Applying the above reasoning over \mathbb{C} instead of \mathbb{F} yields that induction is a bijection between $\text{Irr}(T \mid \vartheta)$ and $\text{Irr}(G \mid \vartheta)$ (this is the well known Clifford correspondence, anyway). That it preserves fields of values and Brauer equivalence classes can now be seen as follows: Suppose $\tau \in \text{Irr}(T \mid \vartheta)$. Let V be a simple $\mathbb{F}T e$ -module whose character contains τ as constituent. Then $\text{End}_{\mathbb{F}T} V$ is a division ring in $[\tau]_{\mathbb{F}}$, and $\mathbb{F}(\tau) \cong \mathbf{Z}(\text{End}_{\mathbb{F}T} V)$. Since we have $\text{End}_{\mathbb{F}T} V \cong \text{End}_{\mathbb{F}G}(V \otimes_{\mathbb{F}T} \mathbb{F}G)$, it follows $[\tau^G]_{\mathbb{F}} = [\tau]_{\mathbb{F}}$ and $\mathbb{F}(\tau^G) \cong \mathbb{F}(\tau)$. Since clearly $\mathbb{F}(\tau^G) \subseteq \mathbb{F}(\tau)$, we have $\mathbb{F}(\tau^G) = \mathbb{F}(\tau)$ as claimed. \square

In general, G_ϑ may be smaller than $G_e = T$. For $\xi \in \text{Irr}(G_\vartheta \mid \vartheta)$, the field $\mathbb{F}(\xi^T)$ is contained in $\mathbb{F}(\xi)$, but may be strictly smaller. If this happens, the Schur index of ξ^T may be bigger than that of ξ .

5.5. Definition. [12, Definition 1.1] Let $K \trianglelefteq G$ and $\vartheta \in \text{Irr } K$. We say that ϑ is *semi-invariant* in G over the field \mathbb{F} , if for every $g \in G$ there is $\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ such that $\vartheta^g = \vartheta^\alpha$. If ϑ is semi-invariant over \mathbb{Q} , then we say it is semi-invariant.

5.6. Lemma. *The following assertions are equivalent:*

- (i) ϑ is semi-invariant over \mathbb{F} in G .
- (ii) The idempotent $e_{(\vartheta, \mathbb{F})}$ is invariant in G .
- (iii) A simple $\mathbb{F}K$ -module whose character contains ϑ as constituent is invariant in G .

Proof. The equivalence between (i) and (ii) is clear.

Let V be a simple $\mathbb{F}K$ -module. The character of V contains ϑ if and only if $Ve_{(\vartheta, \mathbb{F})} = V$ (by Lemma 5.1), and V is determined uniquely up to isomorphism by this property, since $\mathbb{F}Ke_{(\vartheta, \mathbb{F})}$ is artinian simple. The equivalence between (ii) and (iii) follows. \square

5.7. Lemma. *Let $K \trianglelefteq G$ and $\vartheta \in \text{Irr } K$ be semi-invariant over \mathbb{F} . Set $\Gamma = \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$.*

- (a) *For every $g \in G$ there is a unique $\alpha_g \in \Gamma$ such that $\vartheta^{g\alpha_g} = \vartheta$.*
- (b) *The map $g \mapsto \alpha_g$ is a group homomorphism from G into Γ with kernel G_ϑ .*
- (c) *For $g \in G$ and $z \in \mathbf{Z}(\mathbb{F}K)$ we have $\omega_\vartheta(z^g) = \omega_\vartheta(z)^{\alpha_g}$, where*

$$\omega_\vartheta: \mathbf{Z}(\mathbb{F}K) \rightarrow \mathbb{F}(\vartheta)$$

is the central character associated with ϑ .

Proof. Assertions (a) and (b) are proved in a paper of Isaacs [12, Lemma 2.1].

Let $D: K \rightarrow \mathbf{M}_d(\overline{\mathbb{F}})$ be a representation affording ϑ . Then D^g , defined by $D^g(k^g) = D(k)$ for $k \in K$, affords ϑ^g . For $\alpha \in \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ we may define a representation D^α by letting α act on the matrix entries; it is clear that D^α affords ϑ^α . Now ω_ϑ is defined by $D(z) = \omega_\vartheta(z)I$ for $z \in \mathbf{Z}(\mathbb{F}K)$. We may extend α_g to $\alpha \in \text{Aut}(\overline{\mathbb{F}})$. Then

$$\omega_\vartheta(z^g)I = \omega_{\vartheta^g}(z^g)I = D^g(z^g) = D^\alpha(z).$$

If $z \in \mathbf{Z}(\mathbb{F}K)$, then $D^\alpha(z) = D(z)^\alpha = \omega_\vartheta(z)^\alpha I$. Thus (c) follows. \square

6. MAGIC CROSSED REPRESENTATIONS

In this section, we assume the following situation:

6.1. Hypothesis. Let G be a group, $K \trianglelefteq G$ and $H \leq G$ with $G = HK$ and set $L = H \cap K$. Let $\overline{\mathbb{F}}$ be an algebraically closed field and let $\varphi \in \text{Irr}_{\overline{\mathbb{F}}} L$ and $\vartheta \in \text{Irr}_{\overline{\mathbb{F}}} K$ be characters of simple, projective modules over $\overline{\mathbb{F}}L$ respective $\overline{\mathbb{F}}K$ and $\mathbb{F} \subseteq \overline{\mathbb{F}}$ a field such that the following conditions hold:

- (a) $n = (\vartheta_L, \varphi) > 0$.
- (b) $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$.
- (c) For every $h \in H$ there is $\gamma = \gamma_h \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$ such that $\vartheta^{h\gamma} = \vartheta$ and $\varphi^{h\gamma} = \varphi$.

The field may have characteristic $p > 0$, then the conditions imply that φ and ϑ have p -defect zero.

This situation typically arises when ϑ and φ correspond under some “natural” correspondence. We mention two examples:

6.2. Examples.

- (a) Suppose $\varphi \in \text{Irr } L$ and $\vartheta \in \text{Irr } K$ are fully ramified with respect to each other. This means that ϑ vanishes on $K \setminus L$ and $\vartheta_L = n\varphi$ with $n^2 = |K : L|$. Equivalently, $e_\vartheta = e_\varphi$. It is clear that then $\mathbb{Q}(\vartheta) = \mathbb{Q}(\varphi)$. Given H as in Hypothesis 6.1, we see that ϑ is semi-invariant in G if and only if φ is semi-invariant in H . If this is the case, Hypothesis 6.1 is true for $\mathbb{F} = \mathbb{Q}$.
- (b) Let π be a set of primes and suppose that G is π -separable. Let $K = \mathbf{O}_{\pi'}(G)$ and $N/K = \mathbf{O}_\pi(G/K)$. Let P be a Hall π -subgroup of N and set $L = \mathbf{C}_K(P)$. Then for $H = \mathbf{N}_G(P)$ we have $G = HK$ and $L = H \cap K$. Glauberman-Isaacs correspondence defines a bijection between $(\text{Irr}_{\overline{\mathbb{F}}}(K))^P$ and $\text{Irr}_{\overline{\mathbb{F}}} L$. Here, \mathbb{F} may be a field of characteristic zero or of characteristic p with $p \in \pi$. By its naturalness, this correspondence commutes with field and

group automorphisms. Thus if $\vartheta \in (\text{Irr}_{\overline{\mathbb{F}}}(K))^P$ and $\varphi \in \text{Irr}_{\overline{\mathbb{F}}}(L)$ correspond and are semi-invariant in H , then Hypothesis 6.1 holds. We will study a generalization of the case $\pi = \{p\}$ in Section 11.

Now assume that Hypothesis 6.1 holds. As we go along, we will introduce further notation, which is meant to carry through.

6.3. Lemma.

$$i = \sum_{\gamma \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})} e_{\varphi}^{\gamma} e_{\vartheta}^{\gamma}$$

is a H -stable nonzero idempotent in $\mathbb{F}K e_{(\vartheta, \mathbb{F})}$, and we have $e_{(\vartheta, \mathbb{F})} i = i = i e_{(\vartheta, \mathbb{F})}$ and $e_{(\varphi, \mathbb{F})} i = i = i e_{(\varphi, \mathbb{F})}$.

Proof. We have $e_{(\vartheta, \mathbb{F})} = \sum_{\gamma \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} e_{\vartheta}^{\gamma}$. Thus $e_{(\vartheta, \mathbb{F})} i = i = i e_{(\vartheta, \mathbb{F})}$ follows from $e_{\vartheta}^{\gamma} e_{\vartheta}^{\gamma'} = 0$ for $\gamma \neq \gamma' \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. A similar argument shows that $e_{(\varphi, \mathbb{F})} i = i = i e_{(\varphi, \mathbb{F})}$. By assumption, $e_{\varphi} e_{\vartheta} \neq 0$ and thus $i \neq 0$, as $e_{\vartheta} i = e_{\varphi} e_{\vartheta}$. For $h \in H$, we have

$$i^h = \sum_{\gamma} e_{\varphi}^{h\gamma} e_{\vartheta}^{h\gamma} = \sum_{\gamma} e_{\varphi}^{\gamma h} e_{\vartheta}^{\gamma h} = i$$

as desired. \square

As $\mathbb{F}K e_{(\vartheta, \mathbb{F})}$ is simple, it follows that $\mathbb{F}K i \mathbb{F}K = \mathbb{F}K e_{(\vartheta, \mathbb{F})}$ and thus $i \mathbb{F}K i$ and $\mathbb{F}K e_{(\vartheta, \mathbb{F})}$ are Morita equivalent.

6.4. Lemma.

$$\mathbf{Z}(i \mathbb{F}K i) \cong \mathbf{Z}(\mathbb{F}K e_{(\vartheta, \mathbb{F})}) \cong \mathbb{F}(\vartheta) = \mathbb{F}(\varphi) \cong \mathbf{Z}(\mathbb{F}L e_{(\varphi, \mathbb{F})})$$

as fields with H -action.

Proof. Since $\mathbb{F}K e_{(\vartheta, \mathbb{F})} = \mathbb{F}K i \mathbb{F}K$, the map $z \mapsto zi$ is an isomorphism between $\mathbf{Z}(\mathbb{F}K e_{(\vartheta, \mathbb{F})})$ and $\mathbf{Z}(i \mathbb{F}K i)$. It commutes with the action of H , as i is invariant in H .

The central character ω_{ϑ} restricts to an isomorphism $\mathbf{Z}(\mathbb{F}K e_{(\vartheta, \mathbb{F})}) \cong \mathbb{F}(\vartheta)$ and commutes with the action of H by Part (c) of Lemma 5.7. The same reasoning applies to φ . This completes the proof. \square

(Alternatively, one can prove this lemma using Lemma 5.3.)

6.5. Lemma. Set $Z = \mathbf{Z}(i \mathbb{F}K i)$ and let $S = (i \mathbb{F}K i)^L$. Then S is a simple subalgebra of $i \mathbb{F}K i$ with center $Z \cong \mathbb{F}(\vartheta)$, and dimension n^2 over Z , and

$$\mathbf{C}_{i \mathbb{F}K i}(S) = \mathbb{F}L i \cong \mathbb{F}L e_{(\varphi, \mathbb{F})}.$$

Proof. Set $i_0 = e_{\varphi} e_{\vartheta}$. The following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{F}L e_{(\varphi, \mathbb{F})} & \xrightarrow{\cdot i} & i \mathbb{F}K i & \xrightarrow{\subseteq} & \mathbb{F}K e_{(\vartheta, \mathbb{F})} \\ \cdot e_{\varphi} \downarrow & & \cdot i_0 \downarrow & & \cdot e_{\vartheta} \downarrow \\ \mathbb{F}(\vartheta) L e_{\varphi} & \xrightarrow{\cdot i_0} & i_0 \mathbb{F}(\vartheta) K i_0 & \xrightarrow{\subseteq} & \mathbb{F}(\vartheta) K e_{\vartheta}. \end{array}$$

(Note that $i e_{\vartheta} = i e_{\varphi} = i i_0 = i_0$.) By Lemma 5.3, its vertical maps are isomorphisms. The result now follows from Lemma 3.2. \square

As in Section 3, we have that $i \mathbb{F}K i \cong S \otimes_Z \mathbb{F}L e_{(\varphi, \mathbb{F})}$. But now H may act nontrivially on Z , so Z is in general not in the center of $i \mathbb{F}G i$. To be precise, we have the following:

6.6. Remark. $\mathbf{Z}(i \mathbb{F}G i) \cap Z = Z^H$.

Proof. Clear since $i \mathbb{F}G i = \sum_{h \in H} i \mathbb{F}K i h$. \square

What we need is a subalgebra S_0 of S such that $\mathbf{Z}(S_0) = Z^H =: Z_0$ and $S = S_0 Z$. We now add to Hypothesis 6.1 the assumption that there is such a subalgebra S_0 in S . For example, if $S \cong \mathbf{M}_n(Z)$, then $\mathbf{M}_n(Z_0)$ is such a subalgebra in $\mathbf{M}_n(Z)$, and its image in S under some isomorphism is such a subalgebra in S . In this example, S_0 depends on the choice of a particular isomorphism $S \cong \mathbf{M}_n(Z)$.

Even worse, there may be different non-isomorphic algebras S_0 of this type. For example, in $S = \mathbf{M}_2(\mathbb{C})$ with $\mathbb{C}^H = \mathbb{R}$ one can choose

$$S_0 = \mathbf{M}_2(\mathbb{R}) \quad \text{or} \quad S_0 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

The following general result is clear in view of Lemma 2.1.

6.7. Lemma. *Let $Z_0 \leq Z$ be a Galois extension of fields, and let S be a central simple algebra with $\mathbf{Z}(S) = Z$. Suppose $S_0 \leq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S = S_0 Z$. Then $S \cong S_0 \otimes_{Z_0} Z$ and $\text{Gal}(Z/Z_0)$ acts on S via $(sz)^\gamma = sz^\gamma$.*

There is a converse: If $\text{Gal}(Z/Z_0)$ acts on S such that the action extends the natural action of $\text{Gal}(Z/Z_0)$ on $Z = \mathbf{Z}(S)$, then $S_0 = \mathbf{C}_S(\text{Gal}(Z/Z_0))$ is central simple with center Z_0 and $S = S_0 Z$ [10, Lemma 1.2]. We will not need this, however.

We return to the situation of Hypothesis 6.1. Note that, if S_0 is given, we have two actions of H on S : The action by conjugation, and an action ε defined by $(s_0 z)^{\varepsilon(h)} = s_0 z^h$. The second action has kernel H_φ , since the map $H \rightarrow \text{Gal}(Z/Z_0)$ has kernel H_φ .

We have now collected all the ideas necessary to generalize the results of Sections 3 and 4 to the situation of Hypothesis 6.1. In particular, the reader will see that i and S are the correct objects to work with (and not the idempotent $e_{(\vartheta, \mathbb{F})} e_{(\varphi, \mathbb{F})}$, for example).

6.8. Lemma. *Suppose Hypothesis 6.1 and let $S_0 \subseteq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S = S_0 Z$. Define $\varepsilon: H \rightarrow \text{Aut } S$ by $(s_0 z)^{\varepsilon(x)} = s_0 z^x$. For any $x \in H/L$ there is $\sigma(x) \in S$ such that for every $s \in S_0$ we have $s^x = s^{\sigma(x)}$. For $x, y \in H/L$ we have*

$$\sigma(x)^{\varepsilon(y)} \sigma(y) = \alpha(x, y) \sigma(xy) \text{ for some } \alpha(x, y) \in Z^*,$$

and $\alpha \in Z^2(H/L, Z^*)$.

We emphasize that S_0 need not be invariant under H .

Proof. Let $x \in H$. The map $s_0 z \mapsto s_0^x z$ (for $s_0 \in S_0$ and $z \in Z$) yields a well defined Z -algebra automorphism of S . By the Skolem-Noether theorem it is inner. This means that there is $\sigma(x) \in S^*$ such that $s_0^x = s_0^{\sigma(x)}$ for all $s_0 \in S_0$. Since L acts trivially on S , we may choose a map $\sigma: H/L \rightarrow S^*$ such that $s_0^h = s_0^{\sigma(Lh)}$ for all $s_0 \in S_0$.

Note that then for $s_0 \in S_0$ and $z \in Z$, we have

$$(s_0 z)^x = s_0^x z^x = s_0^{\sigma(x)} z^x = (s_0 z^x)^{\sigma(x)} = (s_0 z)^{\varepsilon(x) \sigma(x)}.$$

Thus

$$s_0^{\sigma(xy)} = (s_0^x)^y = (s_0^{\sigma(x)})^y = (s_0^{\sigma(x)})^{\varepsilon(y) \sigma(y)} = s_0^{\sigma(x)^{\varepsilon(y) \sigma(y)}}.$$

Since $\mathbf{C}_S(S_0) = Z$, it follows that $\sigma(x)^{\varepsilon(y) \sigma(y)} = \alpha(x, y) \sigma(xy)$ for some $\alpha(x, y) \in Z^*$. From

$$\left(\sigma(x)^{\varepsilon(y) \sigma(y)} \right)^{\varepsilon(z)} \sigma(z) = \sigma(x)^{\varepsilon(yz)} \left(\sigma(y)^{\varepsilon(z)} \sigma(z) \right)$$

it follows that

$$\alpha(x, y)^z \alpha(xy, z) = \alpha(x, yz) \alpha(y, z).$$

Thus $\alpha \in Z^2(H/L, Z^*)$. □

We may call $\sigma: H/L \rightarrow S$ a “crossed projective representation”. If we want to be more precise, we speak of an ε -crossed projective representation or a projective representation which is crossed with respect to S_0 . Note that $S_0 = \mathbf{C}_S(\varepsilon(H))$, so that S_0 is determined by ε . Conversely, it is clear that ε is determined by S_0 .

If S_0 is fixed, then $\sigma(x)$ is unique up to multiplication with elements of Z , and thus the image of α in $H^2(H/L, Z^*)$ is independent of the particular choice of σ . The image of α in $H^2(H/L, Z^*)$ *does* depend on the choice of S_0 , however. On the positive side, we have:

6.9. Lemma. *The class of α in $H^2(H/L, Z^*)$ depends only on the isomorphism class of S_0 .*

Proof. Suppose S_0 and T_0 are isomorphic. Since $S \cong S_0 \otimes_{Z_0} Z \cong T_0 \otimes_{Z_0} Z$, there is (by the Skolem-Noether theorem) $u \in S^*$ such that $T_0 = (S_0)^u$.

For $h \in H$, define $\delta(h) \in \text{Aut } S$ by $(t_0 z)^{\delta(h)} = t_0 z^h$ for $t_0 \in T_0$ and $z \in Z$. Observe that $s^{\delta(h)} = s^{u^{-1}\varepsilon(h)u}$.

Now for $x \in H/L$, set

$$\tau(x) = [u, \varepsilon(x)]\sigma(x) = u^{-1}u^{\varepsilon(x)}\sigma(x).$$

Then for $t_0 = s_0^u \in T_0$ we have

$$t_0^{\tau(x)} = s_0^{uu^{-1}u^{\varepsilon(x)}\sigma(x)} = (s_0^u)^{\varepsilon(x)\sigma(x)} = (s_0^u)^x = t_0^x$$

and

$$\begin{aligned} \tau(x)^{\delta(y)}\tau(y) &= \left(u^{-1}u^{\varepsilon(x)}\sigma(x)\right)^{u^{-1}\varepsilon(y)u} u^{-1}u^{\varepsilon(y)}\sigma(y) \\ &= u^{-1}u^{\varepsilon(xy)}\sigma(x)^{\varepsilon(y)}\sigma(y) = \alpha(x, y)\tau(xy). \end{aligned}$$

Thus σ and τ define the same cocycle. \square

6.10. Definition. In the situation of Lemma 6.8, we call $\sigma: H/L \rightarrow S$ a *magic* (ε -) crossed representation for the configuration of Hypothesis 6.1 (with respect to S_0), if

- (a) $\sigma(x)^{\varepsilon(y)}\sigma(y) = \sigma(xy)$ for all $x, y \in H/L$ and
- (b) $s^x = s^{\varepsilon(x)\sigma(x)}$ for all $x \in H/L$ and $s \in S$.

It is possible that for one choice of S_0 , there exists a magic crossed representation, while for another choice such a magic crossed representation does not exist. For more details on this question, see [17, pp. 2.37-2.41].

6.11. Theorem. *Assume Hypothesis 6.1, and let $S_0 \subseteq S$ with $S = \mathbf{Z}(S)S_0$ and $\mathbf{Z}(S_0) = \mathbf{Z}(S)^H$. Define $\varepsilon: H/H_\varphi \rightarrow \text{Aut } S$ by $(s_0 z)^{\varepsilon(h)} = s_0 z^h$ for $s \in S_0$ and $z \in \mathbf{Z}(S)$. Let $\sigma: H/L \rightarrow S$ be a magic ε -crossed representation. Then the linear map*

$$\kappa: \mathbb{F}H \rightarrow C = \mathbf{C}_{i\mathbb{F}Gi}(S_0), \quad \text{defined by } h \mapsto h\sigma(Lh)^{-1} \text{ for } h \in H,$$

is an algebra-homomorphism and induces an isomorphism $\mathbb{F}He_{(\varphi, \mathbb{F})} \cong C$.

Proof. Let $c_h = h\sigma(Lh)^{-1}$. It is easy to see that indeed $c_h \in C$. We compute

$$\begin{aligned} c_h c_g &= h\sigma(Lh)^{-1}g\sigma(Lg)^{-1} = hg \left(\sigma(Lh)^{-1}\right)^{\varepsilon(g)\sigma(Lg)} \sigma(Lg)^{-1} \\ &= hg\sigma(Lg)^{-1} \left(\sigma(Lh)^{\varepsilon(g)}\right)^{-1} = hg \left(\sigma(Lh)^{\varepsilon(g)}\sigma(Lg)\right)^{-1} \\ &= hg\sigma(Lhg)^{-1} = c_{hg}. \end{aligned}$$

Thus κ is an algebra homomorphism. From $\sigma(1)^{\varepsilon(1)}\sigma(1) = \sigma(1)$ we see that $\sigma(1) = 1_S = i$, and thus $\kappa(l) = li$ for all $l \in L$. Thus $\kappa(\mathbb{F}Le_{(\varphi, \mathbb{F})}) = \mathbb{F}Li$ and $\kappa(f) = 0$ for

every central primitive idempotent f of $\mathbb{F}L$ different from $e_{(\varphi, \mathbb{F})}$. Now it follows as in the proof of Theorem 3.8 that κ induces an isomorphism from $\mathbb{F}He_{(\varphi, \mathbb{F})}$ onto C . \square

6.12. Corollary. *Assume Hypothesis 6.1 and let $\sigma: H/L \rightarrow S$ be a magic crossed representation, with respect to $S_0 \subseteq S$. Then $i\mathbb{F}Gi \cong S_0 \otimes_{Z_0} \mathbb{F}He_{(\varphi, \mathbb{F})}$. If $S_0 \cong \mathbf{M}_n(Z_0)$, then $i\mathbb{F}Gi \cong \mathbf{M}_n(\mathbb{F}He_{(\varphi, \mathbb{F})})$ and $\mathbb{F}Ge_{(\vartheta, \mathbb{F})}$ and $\mathbb{F}He_{(\varphi, \mathbb{F})}$ are Morita equivalent.*

Proof. All assertions follow from the first. Let $C = \mathbf{C}_{i\mathbb{F}Gi}(S_0)$. Then by Lemma 2.1 we have $i\mathbb{F}Gi \cong S_0 \otimes_{Z_0} C$ (Remember that $Z_0 \subseteq \mathbf{Z}(i\mathbb{F}Gi)$). By Theorem 6.11, $C \cong \mathbb{F}He_{(\varphi, \mathbb{F})}$. The result follows. \square

Suppose $\mathbb{F} \leq \mathbb{C}$. Let us write $\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})$ for the set of all irreducible characters $\chi \in \text{Irr } G$ such that $\chi(e_{(\vartheta, \mathbb{F})}) \neq 0$. (This notation could be used for arbitrary idempotents $e \in \mathbb{C}G$.) Of course,

$$\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})}) = \bigcup_{\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} \text{Irr}(G \mid \vartheta^\alpha).$$

6.13. Theorem. *Assume Hypothesis 6.1 with $\mathbb{F} \leq \mathbb{C}$. Every magic crossed representation σ defines linear isometries*

$$\iota = \iota(\sigma): \mathbb{C}[\text{Irr}(U \mid e_{(\vartheta, \mathbb{F})})] \rightarrow \mathbb{C}[\text{Irr}(U \cap H \mid e_{(\varphi, \mathbb{F})})] \quad (K \leq U \leq G)$$

with Properties (a)–(i) from Theorem 4.3 (where S has to be replaced by S_0 in Property (i)).

Proof. Let us first assume that $\mathbb{F}(\varphi)^H = \mathbb{F}$. Then $Z_0 = \mathbb{F}1_S = \mathbb{F}i$ and $i\mathbb{F}Gi \cong S_0 \otimes_{\mathbb{F}} C$, where $C = \mathbf{C}_{i\mathbb{F}Gi}(S_0)$. The magic crossed representation σ defines an isomorphism $\mathbb{F}He_{(\varphi, \mathbb{F})} \rightarrow C$. By scalar extension, we get an isomorphism $\mathbb{C}He_{(\varphi, \mathbb{F})} \rightarrow \mathbb{C} \otimes_{\mathbb{F}} C$. Note that $\mathbb{C} \otimes_{\mathbb{F}} C = \mathbf{C}_{\mathbb{C} \otimes_{\mathbb{F}}(i\mathbb{F}Gi)}(\mathbb{C} \otimes_{\mathbb{F}} S_0)$. Since $\mathbb{C} \otimes_{\mathbb{F}} S_0$ is central simple, we get isomorphisms

$$\begin{aligned} \text{ZF}(\mathbb{C}Ge_{(\vartheta, \mathbb{F})}, \mathbb{C}) &\xrightarrow[(2.7)]{\text{Res}} \text{ZF}(i\mathbb{C}Gi, \mathbb{C}) \\ &\xrightarrow[(2.4)]{\varepsilon} \text{ZF}(\mathbb{C} \otimes_{\mathbb{F}} C, \mathbb{C}) \xrightarrow[(6.11)]{\kappa^*} \text{ZF}(\mathbb{C}He_{(\varphi, \mathbb{F})}, \mathbb{C}). \end{aligned}$$

As before, for $\chi \in \mathbb{C}[\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})]$ we have $\chi^\iota(h) = \chi(s_0 \sigma(Lh)^{-1}h)$, where $s_0 \in S_0$ is some element with $\text{tr}_{S_0/\mathbb{F}}(s_0) = 1$. The rest of the proof of Theorem 4.3 now carries over verbatim.

Now drop the assumption that $\mathbb{F}(\varphi)^H = \mathbb{F}$ and set $\mathbb{E}_0 = \mathbb{F}(\varphi)^H$. (Thus $\mathbb{E}_0 \cong Z_0 = \mathbf{Z}(S)^H$.) Then

$$\mathbb{E}_0 = \mathbb{F}(\sum_{g \in [G:G_\vartheta]} \vartheta^g) = \mathbb{F}(\sum_{h \in [H:H_\varphi]} \varphi^h),$$

and $\mathbb{F}(\chi)$ contains \mathbb{E}_0 for any $\chi \in \text{Irr}(G \mid e_{(\vartheta, \mathbb{F})}) \cup \text{Irr}(H \mid e_{(\varphi, \mathbb{F})})$. Also φ and ϑ remain semi-invariant over \mathbb{E}_0 .

Observe that

$$e_{(\vartheta, \mathbb{F})} = \sum_{\alpha \in \text{Gal}(\mathbb{E}_0/\mathbb{F})} e_{(\vartheta, \mathbb{E}_0)}^\alpha \quad \text{and} \quad e_{(\varphi, \mathbb{F})} = \sum_{\alpha \in \text{Gal}(\mathbb{E}_0/\mathbb{F})} e_{(\varphi, \mathbb{E}_0)}^\alpha.$$

All the idempotents $e_{(\vartheta, \mathbb{E}_0)}^\alpha$ and $e_{(\varphi, \mathbb{E}_0)}^\alpha$ are invariant in H . Set

$$j = T_{\mathbb{E}_0}^{\mathbb{F}(\vartheta)}(e_\varphi e_\vartheta) = \sum_{\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{E}_0)} (e_\varphi e_\vartheta)^\alpha.$$

Then j is invariant in H and $i = T_{\mathbb{F}}^{\mathbb{E}_0}(j)$. The following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{F}He_{(\varphi, \mathbb{F})} & \xrightarrow{\cdot i} & i\mathbb{F}Gi & \xrightarrow{\subseteq} & \mathbb{F}Ge_{(\vartheta, \mathbb{F})} \\ \cdot e_{(\varphi, \mathbb{E}_0)} \downarrow & & \cdot j \downarrow & & \cdot e_{(\vartheta, \mathbb{E}_0)} \downarrow \\ \mathbb{E}_0He_{(\varphi, \mathbb{E}_0)} & \xrightarrow{\cdot j} & j\mathbb{E}_0Gj & \xrightarrow{\subseteq} & \mathbb{E}_0Ge_{(\vartheta, \mathbb{E}_0)} \end{array}$$

The vertical maps are isomorphisms (by a generalization of Lemma 5.3). It follows that the map $H/L \ni x \mapsto \tau(x) = \sigma(x)j$ is a crossed magic representation $\tau: H/L \rightarrow Sj = (j\mathbb{E}_0Kj)^L$ with respect to S_0j .

Let $\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. Then τ^α defined by $x \mapsto \tau(x)^\alpha = \sigma(x)j^\alpha$ is a magic crossed representation for the configuration of ϑ^α and φ^α . This follows from the fact that τ is magic, or from the above argument with j^α , $e_{(\varphi^\alpha, \mathbb{E}_0)}$ and $e_{(\vartheta^\alpha, \mathbb{E}_0)}$ instead of j , $e_{(\varphi, \mathbb{E}_0)}$ and $e_{(\vartheta, \mathbb{E}_0)}$.

By the first part of the proof, the maps τ^α ($\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$) determine isometries $\iota(\tau^\alpha)$ between $\mathbb{C}[\text{Irr}(G \mid e_{(\vartheta^\alpha, \mathbb{E}_0)})]$ and $\mathbb{C}[\text{Irr}(H \mid e_{(\varphi^\alpha, \mathbb{E}_0)})]$ commuting with field automorphisms over \mathbb{E}_0 .

Choose $\alpha_i \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with

$$\text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F}) = \bigcup_{i=1}^{|\mathbb{E}_0: \mathbb{F}|} \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{E}_0)\alpha_i.$$

Thus

$$\text{Gal}(\mathbb{E}_0/\mathbb{F}) = \{\alpha_i|_{\mathbb{E}_0} \mid i = 1, \dots, |\mathbb{E}_0: \mathbb{F}|\}$$

and

$$\mathbb{C}[\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})] = \bigoplus_i \mathbb{C}[\text{Irr}(G \mid e_{(\vartheta^{\alpha_i}, \mathbb{E}_0)})].$$

We define

$$\iota(\sigma) = \bigoplus_i \iota(\tau^{\alpha_i}): \mathbb{C}[\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})] \rightarrow \mathbb{C}[\text{Irr}(H \mid e_{(\varphi, \mathbb{F})})].$$

If $\alpha|_{\mathbb{E}_0} = \beta|_{\mathbb{E}_0}$, then $\tau^\alpha = \tau^\beta$ and $\iota(\tau^\alpha) = \iota(\tau^\beta)$. Thus $\iota(\sigma)$ is independent of the choice of the α_i . It is clear that $\iota(\sigma)$ commutes with field automorphisms fixing \mathbb{F} . The isometries $\iota(\tau^\alpha)$ preserve Brauer equivalence classes of irreducibles characters over \mathbb{E}_0 . Since $\mathbb{E}_0 \subseteq \mathbb{F}(\chi)$ for all $\chi \in \text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})$, it follows that $\iota(\tau^\alpha)$ preserves Brauer equivalence classes over \mathbb{F} . Thus $\iota(\sigma)$ has Property (i). The other properties are clear. The proof is finished. \square

6.14. Remark. We explain the relation between the last result (respective the more special Theorem 4.3) and Turull's theory of the Brauer-Clifford group [31]. The Brauer-Clifford group of a group X and a commutative X -algebra consists of equivalence classes of certain X -algebras. In the situation of Hypothesis 6.1, there are defined elements $[[\vartheta]]$ and $[[\varphi]]$ of the Brauer-Clifford group $\text{BrClif}(H/L, \mathbb{F}(\vartheta))$ of H/L over $\mathbb{F}(\vartheta)$. The equivalence class $[[S]]$ of the H/L -algebra S also belongs to $\text{BrClif}(H/L, \mathbb{F}(\vartheta))$, and one can show that $[[\vartheta]] = [[S]][[\varphi]]$ in the Brauer-Clifford group [17, Theorem A.35]. In particular, if $[[S]] = 1$, then a result of Turull [31, Theorem 7.12] yields a character bijection as in Theorem 6.13. One can also show that $[[S]] = 1$ if and only if $S \cong \mathbf{M}_n(Z)$ and a magic crossed representation $H/L \rightarrow S$ exists, which is crossed with respect to a subalgebra $S_0 \cong \mathbf{M}_n(Z_0)$.

Our next goal is to exhibit the relation between the results of this section and those of Sections 3 and 4. In Section 3, our standing assumption (Hypothesis 3.1) was that φ and ϑ are invariant in H and that the field \mathbb{F} contains the values of φ and ϑ . Under the assumptions which are in force in this section (Hypothesis 6.1)

this is not the case. However, Hypothesis 3.1 holds for the configuration we get if we replace the group G by G_ϑ , the subgroup H by H_φ and the field \mathbb{F} by $\mathbb{F}(\varphi)$. Thus it makes sense to ask if there is a magic representation $H_\varphi/L \rightarrow (e_\varphi e_\vartheta \mathbb{F}(\varphi) K e_\varphi e_\vartheta)^L$ in the sense of Definition 3.7 for the configuration $(G_\vartheta, H_\varphi, K, L, \vartheta, \varphi)$ over the field $\mathbb{F}(\varphi)$.

6.15. Proposition. *Assume Hypothesis 6.1 and let $\sigma: H/L \rightarrow S$ be a magic crossed representation. Set $i_\varphi = e_\vartheta e_\varphi$. Then*

$$\sigma_\varphi: H_\varphi/L \rightarrow T = (i_\varphi \mathbb{F}(\varphi) K i_\varphi)^L, \quad h \mapsto \sigma_\varphi(h) = \sigma(h) i_\varphi$$

is a magic representation for the configuration $(G_\vartheta, H_\varphi, K, L, \vartheta, \varphi)$ and for $\chi \in \text{Irr}(G_\vartheta \mid \vartheta)$ we have

$$(\chi^G)^{\iota(\sigma)} = (\chi^{\iota(\sigma_\varphi)})^H.$$

Proof. For $s \in S$, we have $s i_\varphi = s e_\vartheta$. That σ_φ is a magic representation follows from Lemma 5.3.

Since $\iota(\sigma)$ commutes with induction of characters, it suffices to show that $\chi^{\iota(\sigma)} = \chi^{\iota(\sigma_\varphi)}$. Choose $s_0 \in S_0$ with $\text{tr}_{S_0/Z_0}(s_0) = 1 = \text{tr}_{S/Z}(s_0)$. Then $\text{tr}_{T/\mathbb{F}(\vartheta)}(s_0 i_\varphi) = 1$. Observe that $e_\vartheta i = i_\varphi$. This yields that for arbitrary $h \in H_\varphi$

$$\begin{aligned} \chi^{\iota(\sigma)}(h) &= \chi(s_0 \sigma(Lh)^{-1} h) = \chi(e_\vartheta s_0 \sigma(Lh)^{-1} h) \\ &= \chi(s_0 i_\varphi \sigma_\varphi(Lh)^{-1} h) = \chi^{\iota(\sigma_\varphi)}(h) \end{aligned}$$

as claimed. \square

This result just means that we get the correspondence $\iota(\sigma)$ by composing the Clifford correspondences associated with ϑ and φ and a correspondence induced by a magic representation. Note that $\iota(\sigma)$ is determined by $\iota(\sigma_\varphi)$ and this property.

Conversely, if the configuration $(G_\vartheta, H_\varphi, K, L, \vartheta, \varphi)$ admits a magic representation τ , we may compose the correspondence $\iota(\tau)$ with the Clifford correspondences between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(G_\vartheta \mid \vartheta)$, respective between $\text{Irr}(H \mid \varphi)$ and $\text{Irr}(H_\varphi \mid \varphi)$. We get then a correspondence between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$. But we do not get compatibility with field automorphisms and Schur indices over the field \mathbb{F} from this argument, and we may have $\mathbb{F}(\vartheta) \not\subseteq \mathbb{F}(\chi)$ for some $\chi \in \text{Irr}(G \mid \vartheta)$.

Finally, let us show that $\iota(\sigma)$ is independent of the particular choice of S_0 :

6.16. Remark. Assume the situation of Theorem 6.13 and let $u \in S^*$. Set $\tau(x) = u^{-1} u^{\varepsilon(x)} \sigma(x)$. Then τ is a magic crossed representation with respect to $T_0 = (S_0)^u$, and $\iota(\sigma) = \iota(\tau)$.

Proof. That τ is magic follows from the proof of Lemma 6.9. Observe that $\tau(x) = u^{-1} \sigma(x) u^x$. Thus

$$\begin{aligned} \chi^{\iota(\tau)}(h) &= \chi(h \tau(Lh)^{-1} (s_0)^u) = \chi(h (u^h)^{-1} \sigma(Lh)^{-1} u (s_0)^u) \\ &= \chi(u^{-1} h \sigma(Lh)^{-1} s_0 u) = \chi(h \sigma(Lh)^{-1} s_0) = \chi^{\iota(\sigma)}(h). \end{aligned}$$

\square

Thus if we view the isomorphism type of S_0 as fixed, we may choose some S_0 without loss of generality. If the subalgebra S_0 is given, a magic representation σ is unique up to multiplication with a map $\lambda: H/L \rightarrow Z^*$ such that $\lambda(x)^y \lambda(y) = \lambda(xy)$ for all $x, y \in H/L$. (In other words, $\lambda \in Z^1(H/L, Z^*)$.) In particular, $\lambda_{H_\varphi} \in \text{Lin}(H_\varphi/L)$. It is not difficult to see that $\chi^{\iota(\lambda\sigma)} = \lambda^{-1} \chi^{\iota(\sigma)}$ for $\chi \in \text{Irr}(G \mid \vartheta)$. As such a χ vanishes on $G \setminus G_\vartheta$, we see that $\iota(\lambda\sigma) = \iota(\sigma)$, if $\lambda_{H_\varphi} = 1$. In particular, $\iota(\lambda\sigma) = \iota(\sigma)$ if $\lambda \in B^1(H/L, Z^*)$, that is, if there is $a \in Z^*$ such that $\lambda(x) = a^{-1} a^x$ for all $x \in H/L$.

For the last remark in this section, note that if σ is a magic crossed representation, then $x \mapsto \text{nr}_{S/Z}(\sigma(x))$ defines an element of $Z^1(H/L, Z^*)$, which we denote simply by $\text{nr}(\sigma)$. (Remember that $\text{nr} = \text{nr}_{S/Z}$ denotes the reduced norm of S with respect to Z .) The following is analogous to Remark 4.4:

6.17. Remark. Let π be the set of prime divisors of n . If there is any magic crossed representation, then there is a magic crossed representation σ such that the class of $\text{nr}(\sigma)$ in $H^1(H/L, Z^*)$ has order a π -number.

Proof. Suppose $\sigma: H/L \rightarrow S$ is given. Let $\lambda = \text{nr}(\sigma) \in Z^1(H/L, Z^*)$. Let b be the π' -part of $\mathfrak{o}(\lambda B^1(H/L, Z^*))$. As $n = \dim \sigma$ is π , there is $r \in \mathbb{Z}$ with $rn + 1 \equiv 0 \pmod{b}$. Then $\text{nr}(\lambda^r \sigma) = \lambda^{rn+1}$ has π -order modulo $B^1(H/L, Z^*)$. \square

Note that $Z^1(H/L, Z^*)$ might be infinite. If it is finite for some reason, we can even get that $\text{nr}(\sigma)$ itself has order a π -number.

7. AN EXAMPLE: COPRIME MULTIPLICITY

We need the following result, which can be found in papers of Dade [7, Theorem 4.4] and Schmid [26, Theorem 2]:

7.1. Proposition. *Let Z/Z_0 be a Galois extension with Galois group Γ and S a central simple Z -algebra such that every $\gamma \in \Gamma$ can be extended to an automorphism of S (as ring). If the Schur index of S is prime to $|\Gamma|$, then there is $S_0 \leq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S = S_0 Z \cong S_0 \otimes_{Z_0} Z$. If $|\Gamma|$ is prime to $\dim_Z S$, then S_0 is unique up to conjugacy (with elements of S^*).*

This can be derived from Teichmüller's work on noncommutative Galois theory, as Schmid [26] has pointed out. (Teichmüller considered simple algebras such that $\text{Aut}_{Z_0} S \rightarrow \text{Gal}(Z/Z_0)$ is surjective. See Eilenberg and MacLane [8] for an exposition and related results.)

7.2. Proposition. *Assume Hypothesis 6.1 with $(n, |H/L|) = 1$. Then there is a magic crossed representation σ such that its reduced norm is in $B^1(H/L, Z^*)$. It yields a canonical correspondence $\iota(\sigma)$ between $\mathbb{C}[\text{Irr}(G \mid e_{(\emptyset, \mathbb{F})})]$ and $\mathbb{C}[\text{Irr}(H \mid e_{(\varphi, \mathbb{F})})]$.*

Proof. As $\Gamma = \text{Gal}(Z/Z_0)$ is a factor group of H/L , it follows that $|\Gamma|$ and $\dim_{\mathbb{F}(\varphi)} S$ are coprime. Thus by Proposition 7.1, there is $S_0 \subseteq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S_0 Z = S$, and S_0 is unique up to inner automorphisms of S . For the moment, fix S_0 . By Lemma 6.8 there is an ε -crossed projective representation with factor set $\alpha \in Z^2(H/L, Z^*)$, say. But as n is coprime to $|H/L|$, it follows that the cohomology class of α is trivial. Thus there exists a magic crossed representation with respect to S_0 .

Since $(n, |H/L|) = 1$, there is a magic crossed representation such that its reduced norm is in $B^1(H/L, Z^*)$ (by Remark 6.17 and since the exponent of $H^1(H/L, Z^*)$ divides $|H/L|$). In particular, for $x \in H_\varphi/L$ we have $\text{nr}(\sigma(x)) = 1$. Let $i_\varphi = e_\varphi e_\vartheta$ and define $\sigma_\varphi(x) = \sigma(x) i_\varphi$ for $x \in H_\varphi/L$ as in Proposition 6.15. Then $\text{nr} \sigma_\varphi(x) = 1$ for $x \in H_\varphi/L$. The magic representation σ_φ is uniquely determined by this condition, and the correspondence $\iota(\sigma)$ is determined by σ_φ (by Proposition 6.15). Thus the correspondence $\iota(\sigma)$ is canonical in the sense that it is independent of the choice of the particular map σ .

It remains to show that $\iota(\sigma)$ is independent of the choice of S_0 . So assume that instead of S_0 we work with S_0^u . Then $\tau(x) = u^{-1} u^{\varepsilon(x)} \sigma(x)$ is a magic crossed representation that yields the same correspondence as σ , by Remark 6.16. Since

$$\text{nr}(\tau(x)) = \text{nr}(u)^{-1} \text{nr}(u)^x \text{nr}(\sigma(x)),$$

it follows that $\text{nr} \tau \in B^1(H/L, Z^*)$ also. \square

7.3. Corollary. *In the situation of Proposition 7.2, there is a uniquely defined character ψ of H_φ/L , and the correspondence ι has the following property: For $\chi \in \text{Irr}(G_\vartheta \mid \vartheta)$, we have*

$$(\chi_{H_\varphi})_\varphi = \psi\chi^\iota \quad \text{and} \quad ((\chi^\iota)^{G_\vartheta})_\vartheta = \overline{\psi}\chi.$$

(We do not claim that ψ is defined by these equations.)

Proof. Let ψ be the magic character of the magic representation $x \mapsto \sigma(x)e_\varphi$. This defines ψ unambiguously. The result follows from Theorem 4.3, in particular (j) and (k). \square

It may be worth pointing out that if we assume Hypothesis 3.1 instead of the more general Hypothesis 6.1, then Proposition 7.2 is an immediate corollary of the results from Sections 3 and 4. One only needs to observe that the cocycle associated with a magic representation must be trivial, since $\dim S$ and $|H/L|$ are coprime. In particular, if one doesn't care about rationality questions and works simply over \mathbb{C} , one can give a rather quick and transparent proof that there is a correspondence between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$, if (ϑ_L, φ) and $|H/L|$ are coprime.

That the Clifford extensions associated with the characters ϑ invariant in G and φ invariant in H are isomorphic was already proved by Dade [2, p. 0.4] in a more general situation, but over an algebraically closed field. Schmid [27] has generalized Dade's result to arbitrary fields, under the additional assumption that the Schur indices of ϑ and φ are coprime to $|H/L|$. On the other hand, both Dade and Schmid work in the more general context of group graded algebras. The description using the magic character ψ seems to be new, however.

It would be nice to have a purely character theoretic description of the correspondences ι . If ψ vanishes nowhere, then χ^ι can be computed from the equation $\chi_{H_\varphi} = \psi\chi^\iota$. In this case, one needs only to know ψ , but not σ or a special element $s_0 \in S_0$ with $\text{tr}(s_0) = 1$, to compute χ^ι . This is true for example if G/K is a p -group:

7.4. Remark. If $x \in H_\varphi/L$ has p -power order, where p is any prime, then $\psi(x) \neq 0$ (in the situation of Proposition 7.2).

Proof. If \mathfrak{P} is a prime ideal of the ring of algebraic integers in \mathbb{C} that lies above p , then $\psi(x) - \psi(1) \in \mathfrak{P}$. Since $\psi(1) = n \notin p\mathbb{Z}$, it follows $\psi(x) \notin \mathfrak{P}$, and so $\psi(x) \neq 0$. \square

The proposition and the corollary apply in particular if $n = 1$. Then $S \cong \mathbb{F}(\varphi)$ and the canonical choice of σ is the trivial map $\sigma(x) = i$ for all x . In particular, for $\chi \in \text{Irr}(G_\vartheta \mid \vartheta)$ we have $(\chi_{H_\varphi})_\varphi = \chi^\iota$. The last equation in fact defines then the correspondence. It follows that χ^ι is the unique element in $\text{Irr}(H_\varphi \mid \varphi)$ with $(\chi_{H_\varphi}, \chi^\iota) \neq 0$ and the correspondence can also be defined by this condition (cf. Proposition 4.6). This fact is known and can be proved just using elementary character theory [14, Lemma 4.1]. Theorem 6.13 also implies that the correspondence preserves Schur indices if $n = 1$: Because then clearly $S_0 \cong \mathbb{F}(\vartheta)^H$ is split. In fact, $\mathbb{F}He_{(\varphi, \mathbb{F})} \cong i\mathbb{F}Gi$ when $n = 1$, where the isomorphism is given by multiplication with i . Note, however, that Hypothesis 6.1 still involves a rather special hypothesis about the fields $\mathbb{F}(\vartheta)$ and $\mathbb{F}(\varphi)$.

8. INDUCTION

8.1. Lemma. *Let G be a finite group, $H \leq U \leq G$ and $K \trianglelefteq G$. Set $N = K \cap U$ and $L = K \cap H$. Let $\vartheta \in \text{Irr } K$, $\eta \in \text{Irr } N$ and $\varphi \in \text{Irr } L$. Assume that the configurations $(G, U, K, N, \vartheta, \eta)$ and $(U, H, N, L, \eta, \varphi)$ fulfill the conditions of Hypothesis 6.1 for the same field \mathbb{F} . Then the conditions of Hypothesis 6.1 also hold for the ‘‘composed’’ configuration $(G, H, K, L, \vartheta, \varphi)$ over \mathbb{F} .*

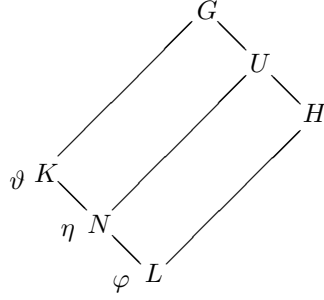


FIGURE 3. A composed basic configuration

Proof. Note that we assume in particular that $G = UK$ and $U = HN$. It follows that $G = HK$ and $L = H \cap K$. (See Figure 3.) By assumption, $(\vartheta_L, \varphi)_L \geq (\vartheta_N, \eta)_N(\eta_L, \varphi)_L > 0$, and $\mathbb{F}(\vartheta) = \mathbb{F}(\eta) = \mathbb{F}(\varphi)$.

Let $h \in H$. By assumption, there is $\gamma_h \in \text{Gal}(\mathbb{F}(\eta)/\mathbb{F})$ such that $\vartheta^{h\gamma_h} = \vartheta$ and $\eta^{h\gamma_h} = \eta$; and there is $\delta_h \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$ such that $\eta^{h\delta_h} = \eta$ and $\varphi^{h\delta_h} = \varphi$. Since only the identity of $\text{Gal}(\mathbb{F}(\eta)/\mathbb{F})$ can fix η^h , it follows that $\delta_h = \gamma_h$. \square

In the situation of the lemma, suppose that we have magic crossed representations σ_1 and σ_2 for the configurations $(G, U, K, N, \vartheta, \eta)$ and $(U, H, N, L, \eta, \varphi)$, respectively. Then we have isometries

$$\begin{aligned} \iota(\sigma_1): \mathbb{C}[\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})] &\rightarrow \mathbb{C}[\text{Irr}(U \mid e_{(\eta, \mathbb{F})})] \quad \text{and} \\ \iota(\sigma_2): \mathbb{C}[\text{Irr}(U \mid e_{(\eta, \mathbb{F})})] &\rightarrow \mathbb{C}[\text{Irr}(H \mid e_{(\varphi, \mathbb{F})})] \end{aligned}$$

which commute with field automorphisms over \mathbb{F} and have the other properties of Theorem 6.13. It follows that

$$\iota(\sigma_1)\iota(\sigma_2): \mathbb{C}[\text{Irr}(G \mid e_{(\vartheta, \mathbb{F})})] \rightarrow \mathbb{C}[\text{Irr}(H \mid e_{(\varphi, \mathbb{F})})]$$

is an isometry as in Theorem 6.13. The question arises if this isometry comes from a magic crossed representation. We try to prove this now.

Define

$$\begin{aligned} i_1 &= \sum_{\gamma} (e_{\vartheta} e_{\eta})^{\gamma}, & i_2 &= \sum_{\gamma} (e_{\eta} e_{\varphi})^{\gamma}, \\ i &= \sum_{\gamma} (e_{\vartheta} e_{\varphi})^{\gamma}, & j &= \sum_{\gamma} (e_{\vartheta} e_{\eta} e_{\varphi})^{\gamma}, \end{aligned}$$

where all sums run over $\gamma \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. Then

$$j = i_1 i_2 = i i_1 = i i_2 = i j = j i.$$

Set

$$S_1 = (i_1 \mathbb{F} K i_1)^N, \quad S_2 = (i_2 \mathbb{F} N i_2)^L \quad \text{and} \quad S = (i \mathbb{F} K i)^L.$$

Then $\mathbf{Z}(S) \cong \mathbf{Z}(S_1) \cong \mathbf{Z}(S_2) \cong \mathbb{F}(\varphi)$. All of the three centers are isomorphic to $\mathbf{Z}(jSj)$ via $z \mapsto zj$.

8.2. Lemma. *The map $s_1 \otimes s_2 \mapsto s_1 s_2 \in S$ defines an isomorphism*

$$S_1 \otimes_{\mathbf{Z}(S)} S_2 \cong jSj.$$

Proof. Clearly, the map is well defined. Since S_1 and S_2 commute, it is a ring homomorphism. The map must be injective since both S_1 and S_2 are central simple over $\mathbf{Z}(S)$. To show that the image is all of jSj , it suffices to show that $\dim_{\mathbf{Z}(S)}(jSj) = n_1^2 n_2^2$, since $n_i^2 = \dim_{\mathbf{Z}(S)}(S_i)$. The isomorphism $\mathbb{F} K e_{(\vartheta, \mathbb{F})} \xrightarrow{e_{\vartheta}}$

$\mathbb{F}(\vartheta)Ke_\vartheta$ takes jSj onto $T := (e_\vartheta e_\eta e_\varphi \mathbb{F}(\vartheta)Ke_\vartheta e_\eta e_\varphi)^L$. To compute the dimension of T over $\mathbb{F}(\vartheta) = \mathbb{E}$, we may assume that \mathbb{E} is a splitting field of K , N and L . Let V be a $\mathbb{E}K$ -module affording ϑ . Then $Ve_\eta \cong n_1 W$ as $\mathbb{E}N$ -module, where W affords η , and $Ve_\eta e_\varphi \cong n_1 n_2 X$ as $\mathbb{E}L$ -module, where X affords φ . Thus $T = (e_\eta e_\varphi \mathbb{E}Ke_\vartheta e_\eta e_\varphi)^L \cong \text{End}_{\mathbb{E}L}(Ve_\eta e_\varphi) \cong \mathbf{M}_{n_1 n_2}(\mathbb{E})$. Therefore $\dim_{\mathbb{E}}(T) = (n_1 n_2)^2$ as claimed. \square

Now let $Z = \mathbf{Z}(S)$ and set $Z_0 = Z^H$, the subfield fixed by the action of H . Our assumption that there are crossed magic representations σ_i includes the assumption that the algebras S_i are obtained by scalar extension from central simple Z_0 -algebras S_{10} and S_{20} , say. The isomorphism of Lemma 8.2 sends $S_{10} \otimes_{Z_0} S_{20}$ onto a central simple Z_0 -algebra T_0 , such that $jSj \cong T_0 \otimes_{Z_0} Z$. Thus the algebra jSj is obtained by scalar extension from Z_0 , and so is the algebra class of jSj in the Brauer group of $\mathbf{Z}(S)$. Of course, S belongs to the same class as jSj . However, it doesn't follow that S itself is obtained by scalar extension from a Z_0 -algebra. (There exist examples to the end that some algebra S can not be obtained by scalar extension, while $\mathbf{M}_k(S)$ can, for some $k > 1$ [8, §14].) Thus we make the following assumption:

8.3. Hypothesis. There are central simple Z_0 -algebras $S_{10} \subseteq S_1$, $S_{20} \subseteq S_2$ and $S_0 \subseteq S$, such that

$$S_1 \cong S_{10} \otimes_{Z_0} Z, \quad S_2 \cong S_{20} \otimes_{Z_0} Z, \quad S \cong S_0 \otimes_{Z_0} Z \quad \text{and} \quad S_{10} S_{20} \subseteq S_0.$$

Moreover, there are magic crossed representations

$$\sigma_1: U/N \rightarrow S_1 \quad \text{and} \quad \sigma_2: H/L \rightarrow S_2$$

which are crossed over S_{10} and S_{20} , respectively.

Thus if we view S_{10} and S_{20} as given, we search for a central simple Z_0 -algebra S_0 that contains both $S_{10}j$ and $S_{20}j$. This is somewhat more than only to assume that S is obtained by scalar extension from a Z_0 -algebra.

8.4. Remark. Suppose $S_{i0} \subseteq S_i$ and σ_i are given. Then Hypothesis 8.3 holds in each of the following cases:

- (a) $Z = Z_0$ (that is, ϑ , η and φ are invariant in H).
- (b) S_{10} and S_{20} are matrix rings over Z_0 and S is a matrix ring over Z .
- (c) $j = i$ or, equivalently, $n = n_1 n_2$.
- (d) n_1 , n_2 and n are all coprime to $|Z : Z_0|$.

(In the last case the result follows from Proposition 7.1.)

We are now able to state the main result of this section:

8.5. Proposition. *In the situation of Lemma 8.1, assume Hypothesis 8.3. Then there is a magic crossed representation*

$$\sigma: H/L \rightarrow S = (e_\varphi \mathbb{F}Ke_\vartheta e_\varphi)^L$$

such that

$$\iota(\sigma) = \iota(\sigma_1)\iota(\sigma_2).$$

For this σ , we have $\sigma(h)j = \sigma_1(h)\sigma_2(h)$.

Proof. Set $T_0 = S_{10}S_{20}$ and $T = jSj$. Define $\tau(h) = \sigma_1(h)\sigma_2(h) \in T$. Then for every $t_0 = s_{10}s_{20} \in T_0$ we have

$$t_0^{\tau(h)} = s_{10}^{\sigma_1(h)} s_{20}^{\sigma_2(h)} = s_{10}^h s_{20}^h = t_0^h,$$

since S_{1j} and S_{2j} commute with each other.

As before, define $\varepsilon(h) \in \text{Aut } S$ by $(s_0 z)^{\varepsilon(h)} = s_0 z^{\varepsilon(h)}$. Then

$$\begin{aligned} \tau(x)^{\varepsilon(y)} \tau(y) &= \sigma_1(x)^{\varepsilon(y)} \sigma_1(y) \sigma_2(x)^{\varepsilon(y)} \sigma_2(y) \\ &= \sigma_1(xy) \sigma_2(xy) = \tau(xy). \end{aligned}$$

Again, the first equality holds since $S_1 j$ and $S_2 j$ commute, and the second follows since $S_{10}, S_{20} \subseteq S_0$ and σ_1 and σ_2 are crossed over S_{10} and S_{20} , respectively.

By Lemma 6.8, a projective crossed representation $\sigma: H/L \rightarrow S$ exists, such that $s_0^h = s_0^{\sigma(h)}$ for all $s_0 \in S_0$. In particular, this holds for elements of T_0 . The idempotent $j = ji$ is H -invariant by assumption, so that $\sigma(h)$ centralizes j for every $h \in H$. Thus $j\sigma(h) = \sigma(h)j \in T$, and this element is invertible in T . For $t_0 \in T_0$ we have thus $t_0^{j\sigma(h)} = t_0^{\sigma(h)} = t_0^h = t_0^{\tau(h)}$. As $\mathbf{C}_T(T_0) = Zj$, it follows that $j\sigma(h) = \lambda_h \tau(h)$ for some $\lambda_h \in Z$. Therefore σ is projectively equivalent with a representation.

From now on, assume that $\sigma: H/L \rightarrow S$ is such that $j\sigma(h) = \tau(h) = \sigma_1(h)\sigma_2(h)$. We want to show that $\iota(\sigma) = \iota(\sigma_1)\iota(\sigma_2)$. Choose $s_i \in S_{i0}$ with $\text{tr}_{S_{i0}/Z_0}(s_i) = 1$. Then

$$\text{tr}_{S_0/Z_0}(s_1 s_2) = \text{tr}_{T_0/Z_0}(s_1 s_2) = \text{tr}_{S_{10}/Z_0}(s_1) \text{tr}_{S_{20}/Z_0}(s_2) = 1.$$

Let $\chi \in \mathbb{C}[\text{Irr}(G | e_{(\emptyset, \mathbb{F})})]$. Let $u \in U$ and $\alpha \in \mathbb{F}N$, and write $\alpha = \sum_{n \in N} \alpha_n n$ with $\alpha_n \in \mathbb{F}$. Since $N \leq \ker \sigma_1$, we have

$$\begin{aligned} \chi^{\iota(\sigma_1)}(\alpha u) &= \sum_{n \in N} \alpha_n \chi(s_1 \sigma_1(nu)^{-1} nu) = \sum_{n \in N} \alpha_n \chi(s_1 \sigma_1(u)^{-1} nu) \\ &= \chi(s_1 \sigma_1(u)^{-1} \alpha u). \end{aligned}$$

Using this with $\alpha = s_2 \sigma_2(h)^{-1}$, where $h \in H$ is arbitrary, we get

$$\begin{aligned} \chi^{\iota(\sigma_1)\iota(\sigma_2)}(h) &= \chi^{\iota(\sigma_1)}(s_2 \sigma_2(h)^{-1} h) \\ &= \chi(s_1 \sigma_1(h)^{-1} s_2 \sigma_2(h)^{-1} h) \quad (\text{see above}) \\ &= \chi(s_1 s_2 \tau(h)^{-1} h) \\ &= \chi(s_1 s_2 \sigma(h)^{-1} h) \quad (\text{as } s_1 s_2 \in jSj) \\ &= \chi^{\iota(\sigma)}(h) \quad (\text{as } \text{tr}_{S_0/Z_0}(s_1 s_2) = 1) . \quad \square \end{aligned}$$

9. LIFTING MAGIC REPRESENTATIONS

In this section, \mathbb{K} denotes a field complete with respect to a discrete valuation $\nu: \mathbb{K} \rightarrow \mathbb{Z}$, with valuation ring A and with a perfect residue class field $\mathbb{F} = A/J(A)$ of characteristic $p > 0$. The letter π denotes a prime element of A , so that $J(A) = A\pi$.

We need the following lemma which is probably well known.

9.1. Lemma. *Let X be a group that acts on A and $\alpha \in Z^2(X, 1 + A\pi)$. Then the order of the cohomology class of α (in $H^2(X, 1 + A\pi)$, and thus in $H^2(X, A^*)$) is a p -number (or ∞ , if $|X|$ is not finite).*

Proof. It suffices to show that if the cohomology class of α has p' -order, then α is a coboundary. So suppose that α is a cocycle such that α^k is a coboundary for some p' -number k , say. Thus $\alpha(x, y)^k = \lambda(x)^y \lambda(y) \lambda(xy)^{-1}$ for some $\lambda: X \rightarrow 1 + A\pi$.

We will construct a sequence of maps $\mu_n: X \rightarrow 1 + A\pi$ ($n = 1, 2, \dots$) such that

$$\begin{aligned} \alpha(x, y) \mu_n(xy) &\equiv \mu_n(x)^y \mu_n(y) \pmod{\pi^n} \quad \text{and} \\ \mu_{n+1}(x) &\equiv \mu_n(x) \pmod{\pi^n} \end{aligned}$$

for all n . It follows that $\mu(x) = \lim_{n \rightarrow \infty} \mu_n(x)$ exists, and that

$$\alpha(x, y) = \mu(x)^y \mu(y) \mu(xy)^{-1},$$

as we want to show.

Choose $a, b \in \mathbb{Z}$ with $ak + bp = 1$. We define μ_n recursively by

$$\mu_1(x) = 1_A, \quad \mu_{n+1}(x) = \lambda(x)^a \mu_n(x)^{bp} \quad (x \in X).$$

We use induction to prove the above properties: By assumption, we have $\alpha(x, y) \equiv 1 \pmod{\pi}$ for all $x, y \in X$. Also $\mu_2(x) = \lambda(x)^a \equiv 1 = \mu_1(x) \pmod{\pi}$ since $\lambda(x) \in 1 + A\pi$ by assumption. From $\mu_{n+1}(x) \equiv \mu_n(x) \pmod{\pi^n}$ it follows $\mu_{n+1}(x)^p \equiv \mu_n(x)^p \pmod{\pi^{n+1}}$ and thus

$$\begin{aligned} \mu_{n+2}(x) &= \lambda(x)^a \mu_{n+1}(x)^{bp} \\ &\equiv \lambda(x)^a \mu_n(x)^{bp} \pmod{\pi^{n+1}} \\ &= \mu_{n+1}(x). \end{aligned}$$

Assuming that $\alpha(x, y) \equiv \mu_n(x)^y \mu_n(y) \mu_n(xy)^{-1} \pmod{\pi^n}$ by induction, we get

$$\begin{aligned} \alpha(x, y) &= \alpha(x, y)^{ak+bp} = (\lambda(x)^y \lambda(y) \lambda(xy)^{-1})^a \alpha(x, y)^{bp} \\ &\equiv (\lambda(x)^y \lambda(y) \lambda(xy)^{-1})^a (\mu_n(x)^y \mu_n(y) \mu_n(xy)^{-1})^{bp} \pmod{\pi^{n+1}} \\ &= \mu_{n+1}(x)^y \mu_{n+1}(y) \mu_{n+1}(xy)^{-1}. \end{aligned}$$

The proof is finished. \square

For convenience, we now fix notation to be used in the rest of this section.

9.2. Hypothesis. Let A be a complete discrete valuation ring with quotient field \mathbb{K} of characteristic zero and residue class field $A/A\pi = \mathbb{F}$ of characteristic $p > 0$. Suppose we are in the situation of Hypothesis 6.1 with \mathbb{K} instead of \mathbb{F} . Recall that $K \trianglelefteq G$ and $H \leq G$ with $G = KH$ and $L = H \cap K$, and that $\vartheta \in \text{Irr } K$ and $\varphi \in \text{Irr } L$ are such that $\mathbb{K}(\vartheta) = \mathbb{K}(\varphi)$, and for every $h \in H$ there is $\gamma_h \in \text{Gal}(\mathbb{K}(\vartheta)/\mathbb{K})$ such that $\vartheta^{\gamma_h} = \vartheta$ and $\varphi^{\gamma_h} = \varphi$.

Let $\mathbb{L} = \mathbb{K}(\vartheta)$ and let B be the integral closure of A in \mathbb{L} . It is well known that ν extends uniquely to a valuation on \mathbb{L} and that B is the corresponding valuation ring [28, Chap. II, § 2]. Let $\mathbb{E} = B/J(B)$ be the residue class field of \mathbb{L} .

In addition, we assume that ϑ and φ have p -defect zero.

It follows that $e_\vartheta \in BK$ and $e_\varphi \in BL$. Moreover, ϑ and φ vanish on elements of order divisible by p [15, p. 8.17]. Thus the values of ϑ and φ are contained in $\mathbb{Q}(\varepsilon)$, where ε is a primitive m -th root of unity with m not divisible by p . It follows that \mathbb{L} is unramified over \mathbb{K} [28, Chap. IV, § 4]. Thus $|\mathbb{L} : \mathbb{K}| = |\mathbb{E} : \mathbb{F}|$ and $J(B) = B\pi$. The canonical homomorphism $\text{Gal}(\mathbb{L}/\mathbb{K}) \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})$ is an isomorphism.

The canonical epimorphism $\bar{\cdot} : B \rightarrow \mathbb{E}$ extends naturally to BG , and we denote this extension also by $\bar{\cdot}$. Since we will work mostly with AG and $\mathbb{F}G$, we emphasize that we also use the symbol $\bar{\cdot}$ to denote the restriction $\bar{\cdot} : AG \rightarrow \mathbb{F}G$, which is an epimorphism from AG onto $\mathbb{F}G$.

Suppose there is a magic crossed representation $\sigma : H/L \rightarrow (\bar{i}\mathbb{F}K\bar{i})^L$, where $i = \sum_{\gamma \in \text{Gal}(\mathbb{L}/\mathbb{K})} e_\vartheta e_\varphi$. We wish to show that σ lifts to a crossed magic representation $\hat{\sigma} : H/L \rightarrow (iAKi)^L \subseteq (i\mathbb{K}Ki)^L$.

Before we do this, we observe the following:

9.3. Lemma. $(\bar{i}\mathbb{F}K\bar{i})^L \cong \mathbf{M}_n(\mathbb{E})$ and $(iAKi)^L \cong \mathbf{M}_n(B)$.

Proof. The first isomorphism follows since $\mathbf{Z}((\bar{i}\mathbb{F}K\bar{i})^L) \cong \mathbb{E}$ and \mathbb{F} has characteristic $p > 0$: Because then $\mathbb{F}K\bar{e}_{(\vartheta, \mathbb{K})}$ and $\mathbb{F}L\bar{i} \cong \mathbb{F}L\bar{e}_{(\varphi, \mathbb{K})}$ are matrix rings over \mathbb{E} , and thus $(\bar{i}\mathbb{F}K\bar{i})^L = \mathbf{C}_{\bar{i}\mathbb{F}K\bar{i}}(\mathbb{F}L\bar{i})$ must be a matrix ring, too.

Since A is complete, we may lift idempotents, and thus the matrix units, to $(iAKi)^L$ [20, Proposition 21.34]. The second isomorphism follows. \square

Next we show that Lemma 6.8 extends to valuation rings. Set $\Sigma = (iAKi)^L$ and $S = (i\mathbb{K}Ki)^L$. The isomorphism $\mathbf{Z}(S) \cong \mathbb{L}$ of Lemma 6.4 makes S into an \mathbb{L} -algebra. Restriction to B yields an isomorphism $B \cong \mathbf{Z}(\Sigma)$, so we may view Σ as a B -algebra. These isomorphisms respect the action of H . Let $\mathbb{L}_0 = \mathbb{L}^H$, the subfield of elements fixed by H , and set $B_0 = B \cap \mathbb{L}_0 (= B^H)$.

By Lemma 9.3, Σ contains a set of matrix units, E (say). Let Σ_0 be the B_0 -subalgebra generated by E , so that $\Sigma_0 \cong \mathbf{M}_n(B_0)$ and $\Sigma \cong \Sigma_0 \otimes_{B_0} B$. Similarly, let S_0 be the \mathbb{L}_0 -subalgebra of S generated by E . Then clearly $S_0 \cong \Sigma_0 \otimes_{B_0} \mathbb{L}_0$.

The identification of S with a matrix ring over \mathbb{L} yields an action of $\text{Gal}(\mathbb{L}/\mathbb{K})$ on S ; since H acts on \mathbb{L} , we get an action of H on S which we denote by ε . Thus, for $s_0 \in S_0$ and $z \in \mathbf{Z}(S)$, we have $(s_0 z)^{\varepsilon(h)} = s_0 z^h$. This action maps Σ into Σ .

Since Σ is a matrix ring over the local ring $\mathbf{Z}(\Sigma) \cong B$, every automorphism of Σ centralizing $\mathbf{Z}(\Sigma)$ is inner [23, Chap. 2, Theorem 4.8]. This applies to $s \mapsto s^{\varepsilon(h)^{-1}h}$. Thus there is, for every $h \in H$, an element $\sigma(h) \in \Sigma^*$ such that $s_0^h = s_0^{\sigma(h)}$ for all $s_0 \in \Sigma_0$ (and then in fact for all $s_0 \in S_0$). This yields a crossed projective representation $\sigma: H/L \rightarrow \Sigma^*$. We have thus proved:

9.4. Lemma. *In the situation of Hypothesis 9.2, there is a crossed projective representation $\sigma: H/L \rightarrow (iAKi)^L = \Sigma$ such that $s^h = s^{\sigma(Lh)}$ for all $s \in \Sigma_0$, where $\mathbf{M}_n(B^H) \cong \Sigma_0 \subseteq (iAKi)^L$.*

9.5. Theorem. *In the situation of Hypothesis 9.2, suppose that there is a magic crossed representation $\sigma: H/L \rightarrow (\bar{i}\mathbb{F}K\bar{i})^L$. Then for every p' -subgroup $V/L \leq H/L$ there is a magic crossed representation $\hat{\sigma}: V/L \rightarrow (iAKi)^L$ lifting $\sigma_{V/L}$ (that is, $\hat{\sigma}(x) = \sigma(x)$ for $x \in V/L$). If $n = (\vartheta_L, \varphi) \not\equiv 0 \pmod{p}$, then there is a magic crossed representation $\hat{\sigma}: H/L \rightarrow (iAKi)^L$ lifting σ .*

Proof. Let $\hat{\sigma}: H/L \rightarrow \Sigma = (iAKi)^L$ be a crossed projective representation, which exists by Lemma 9.4.

Since $\hat{\sigma}(h) \in \Sigma^*$, reduction modulo π yields a crossed projective representation

$$h \mapsto \overline{\hat{\sigma}(h)} \in \bar{\Sigma} = (\bar{i}\mathbb{F}K\bar{i})^L.$$

Clearly, $t^{\overline{\hat{\sigma}(h)}} = t^h$ for $t \in \bar{\Sigma}_0 \cong \mathbf{M}_n(\mathbb{F}^H)$. After multiplying $\hat{\sigma}$ with a suitable factor from $\mathbf{Z}(\Sigma)$, we may assume that $\overline{\hat{\sigma}(h)} = \sigma(h)$ for $h \in H$. Let $\alpha \in Z^2(H/L, B^*)$ be the cocycle associated with $\hat{\sigma}$. Then α has values in $1 + B\pi$, since σ is multiplicative. By Lemma 9.1, the cohomology class of α has p -order. In particular, $\alpha_{V/L} \sim 1$ for any p' -group V/L . Thus $\hat{\sigma}_{V/L}$ is projectively equivalent with a crossed representation. If $n \not\equiv 0 \pmod{p}$, then it follows $\alpha \sim 1$, since the class of α has order dividing n . The proof is finished. \square

10. REDUCING MAGIC REPRESENTATIONS MODULO A PRIME

Assume Hypothesis 9.2. As before, let $i = \sum_{\gamma \in \text{Gal}(\mathbb{L}/\mathbb{K})} (e_{\vartheta} e_{\varphi})^{\gamma}$ and $S = (i\mathbb{K}Ki)^L$. Our aim here is to show that if there is a magic representation $\sigma: H/L \rightarrow S$, then not only $\mathbb{K}Ge_{(\vartheta, \mathbb{K})}$ and $\mathbb{K}He_{(\varphi, \mathbb{K})}$ are Morita equivalent, but also $AGe_{(\vartheta, \mathbb{K})}$ and $AHe_{(\varphi, \mathbb{K})}$.

There is a quite general result of Broué of this kind [1], but verifying the premises of Broué's result is nearly the same amount of work as proving the desired result directly.

10.1. Lemma. $AKe_{(\vartheta, \mathbb{K})} = AKiAK$.

Proof. As $\mathbb{F}K\overline{e_{(\vartheta, \mathbb{K})}}$ is simple, we have $\mathbb{F}K\overline{e_{(\vartheta, \mathbb{K})}} = \mathbb{F}K\bar{i}\mathbb{F}K$. Thus

$$AKe_{(\vartheta, \mathbb{K})} = AKiAK + \pi AKe_{(\vartheta, \mathbb{K})}.$$

By Nakayama's lemma, the result follows. \square

It follows that $AGe_{(\vartheta, \mathbb{K})}$ and $iAGi$ are Morita equivalent. We now assume that a magic crossed representation $\sigma: H/L \rightarrow S \cong \mathbf{M}_n(\mathbb{L})$ exists. We know that then $i\mathbb{K}Gi \cong \mathbf{M}_n(\mathbb{K}He_{(\varphi, \mathbb{K})})$, and we want to show that the same is true if we replace \mathbb{K} by A . We have seen in the last section that $\Sigma = (iAKi)^L \cong \mathbf{M}_n(B)$. Choose $\Sigma_0 \subseteq \Sigma$ with $\Sigma_0 \cong \mathbf{M}_n(B^H)$ and let $S_0 = \mathbb{K}\Sigma_0 \cong \mathbb{K} \otimes_A \Sigma_0$. Then $iAGi \cong \mathbf{M}_n(\Gamma)$, where $\Gamma = \mathbf{C}_{iAGi}(\Sigma_0)$. It is clear that $\Gamma = AG \cap C$, where $C = \mathbf{C}_{i\mathbb{K}Gi}(S_0)$. We have to show that the isomorphism of Theorem 6.11 sends $AHe_{(\varphi, \mathbb{K})}$ onto Γ . We first show that $\sigma(H) \subseteq \Sigma = (iAKi)^L$.

10.2. Lemma. *In the situation of Hypothesis 9.2, suppose that there is a magic crossed representation $\sigma: H/L \rightarrow S = (i\mathbb{K}Ki)^L$. Then $\sigma(H/L) \subseteq \Sigma = (iAKi)^L$.*

Proof. By Lemma 9.4, there also exists a projective crossed representation $\hat{\sigma}: H/L \rightarrow \Sigma$. For $h \in H$, there is $\lambda \in \mathbf{Z}(S) \cong \mathbb{L}$ such that $\hat{\sigma}(h) = \lambda\sigma(h)$, and thus $\lambda\sigma(h) \in \Sigma^*$. This means that also $(\lambda\sigma(h))^{-1} = \lambda^{-1}\sigma(h^{-1}) \in \Sigma^*$. As λ or λ^{-1} is in $\mathbf{Z}(\Sigma) \cong B$, it follows that $\sigma(h^{-1})$ or $\sigma(h)$ is in Σ . But as σ is a crossed representation and h has finite order, both are in Σ . Thus $\sigma(H) \subseteq \Sigma$ as claimed. \square

Remark. Completeness of \mathbb{K} was not used in the previous proof, and in fact the lemma is true for any ring A integrally closed in its quotient field, and such that $i \in AK$. This follows because such a ring is the intersection of all the valuation rings containing it.

10.3. Lemma. *Keep the notation above and assume that $\sigma: H/L \rightarrow S$ is magic. Then the homomorphism κ of Theorem 6.11 maps AHe_φ onto Γ .*

Proof. From Lemma 10.2 it follows that $c_h = h\sigma(Lh)^{-1} \in C \cap AG = \Gamma$. Therefore $(AHe_{(\varphi, \mathbb{K})})\kappa \subseteq \Gamma$. As in the proof of Theorem 6.11, we see that $\Gamma = \bigoplus_{t \in T} \Gamma_1 c_t$, where $\Gamma_1 = \Gamma \cap AK$ and T is a set of representative of the cosets of L in H . Thus it suffices to show that $(AHe_{(\varphi, \mathbb{K})})\kappa = \Gamma_1$. We know that $(AHe_{(\varphi, \mathbb{K})})\kappa = ALi$ and that $\Gamma_1 = \mathbb{K}Li \cap iAKi$. Clearly $ALi \subseteq \Gamma_1$. But since $ALi \cong ALe_{(\varphi, \mathbb{K})} \cong \mathbf{M}_n(B)$, the subring ALi must be a maximal A -order of $\mathbb{K}Li \cong \mathbf{M}_n(B)$ [25, Theorem 8.7]. Since $\Gamma_1 \subseteq iAKi$ is an A -order, the proof now follows. \square

From what we have done so far it follows that, if a magic crossed representation $\sigma: H/L \rightarrow S = (i\mathbb{K}Ki)^L$ exists in the situation of Hypothesis 9.2, then AHe_φ and $AGe_{(\vartheta, \mathbb{K})}$ are Morita equivalent, and the equivalence induces the character bijection of Theorem 6.13. The equivalence can be described more concretely: First, choose an primitive idempotent $j \in \Sigma = (iAKi)^L$. Then we have $jAGj = jiAGij \cong \Gamma \cong AHe_\varphi$, where an isomorphism from AHe_φ onto $jAGj$ is induced by the map sending $h \in H$ to $jh\sigma(Lh)^{-1} = h\sigma(Lh)^{-1}j = jh\sigma(Lh)^{-1}j$. Also we have $i = 1_S \in \Sigma j \Sigma$ and $e_{(\vartheta, \mathbb{K})} \in AGiAG = AGe_{(\vartheta, \mathbb{K})}$, so that $AGjAG = AGe_{(\vartheta, \mathbb{K})}$. The idempotent j is thus full in $AGe_{(\vartheta, \mathbb{K})}$ and we have a Morita equivalence between $AGe_{(\vartheta, \mathbb{K})}$ and $jAGj = jAGe_{(\vartheta, \mathbb{K})}j$ sending an $AGe_{(\vartheta, \mathbb{K})}$ -module V to Vj and an $jAGj$ -module U to $U \otimes_{jAGj} jAG$ [19, Example 18.30]. Since $jAGj \cong AHe_\varphi$, this gives also an Morita equivalence between $AGe_{(\vartheta, \mathbb{K})}$ and AHe_φ . We now have proved:

10.4. Theorem. *Assume Hypothesis 9.2 and let $\sigma: H/L \rightarrow S$ be a magic representation. Then there is an idempotent $j \in (iAKi)^L = \Sigma$ such that $AHe_{(\varphi, \mathbb{K})} \cong jAGj$ via the map defined by $h \mapsto jh\sigma(Lh)^{-1}$, and $AGjAG = AGe_{(\vartheta, \mathbb{K})}$. The rings $AHe_{(\varphi, \mathbb{K})}$ and $AGe_{(\vartheta, \mathbb{K})}$ are Morita equivalent.*

Remark. The Morita equivalence is graded in the sense of [21, 22].

11. ABOVE THE GLAUBERMAN CORRESPONDENCE

In this section, we need some notation and results from the theory of G -algebras, which we review shortly. Let A be an algebra, and suppose the group G acts on A . (Shortly, A is a G -algebra.) Remember that for subgroups $Q \leq P \leq G$ we have a trace map $T_Q^P: A^Q \rightarrow A^P$ given by $T_Q^P(a) = \sum_{x \in [P:Q]} a^x$ (where $[P:Q]$ denotes a set of representatives of the left cosets of Q in P), and that $A_Q^P := T_Q^P(A^Q)$ is an ideal of A^P [29, § 11].

Now let A be an algebra over a field \mathbb{E} of characteristic $p > 0$. We write $A(P)$ for the factor algebra $A^P/(A_{<P}^P)$, where $A_{<P}^P$ denotes the ideal $\sum_{Q < P} A_Q^P$, the sum running over proper subgroups of P . The canonical homomorphism $\text{br}_P = \text{br}_P^A: A^P \rightarrow A(P)$ is called the Brauer homomorphism. If $A(P) \neq 0$, then P is a p -group.

In the special case where $A = \mathbb{E}K$ and P acts on the group K , we may (and do) identify $A(P)$ with $\mathbb{E} \mathbf{C}_K(P)$; the Brauer homomorphism becomes the usual projection $(\mathbb{E}K)^P \rightarrow \mathbb{E} \mathbf{C}_K(P)$.

If $e \in A^G$ is a primitive idempotent, then the minimal subgroups among the subgroups $D \leq G$ for which $e \in A_D^G$, are conjugate in G and are called *defect groups* of e . Defect groups are also characterized as maximal subgroups subject to $\text{br}_D(e) \neq 0$ [29, § 18].

We describe now the situation we study in this section. (It is the same situation as studied by Dade [6].) Let G be a finite group with normal subgroups $K \leq M$ such that M/K is a p -group. Let $\vartheta \in \text{Irr } K$ be invariant in M and semi-invariant in G and suppose that ϑ has p -defect zero. Observe that then the coefficients of the central idempotent $e_\vartheta \in \mathbb{Q}(\vartheta)G$ are contained in the valuation ring $\mathbb{Z}_p[\vartheta]$ (which is unramified over \mathbb{Z}_p). Let $\mathbb{E} = \mathbb{Z}_p[\vartheta]/p\mathbb{Z}_p[\vartheta]$ be the residue class field. Write \bar{e}_ϑ for the image of e_ϑ in $\mathbb{E}K$ under the homomorphism $\mathbb{Z}_p[\vartheta]K \rightarrow \mathbb{E}K$. Then $\bar{e}_\vartheta \in \mathbf{Z}(\mathbb{E}K)$ is a block idempotent of $\mathbb{E}K$, and we have $\bar{e}_\vartheta = T_1^K(a)$ for some $a \in \mathbb{E}K\bar{e}_\vartheta$, since ϑ has p -defect zero.

One may view $\mathbb{E}K$ as an M -algebra. Let $P \leq M$ be a defect group of the idempotent $\bar{e}_\vartheta \in (\mathbb{E}K)^M$. Then $M = KP$ and $K \cap P = 1$. Set $H = \mathbf{N}_G(P)$. Since ϑ is semi-invariant in G and Galois conjugate blocks have the same defect groups, it follows from the Frattini argument that $G = HK$. Set $L = H \cap K$, so that $L = \mathbf{C}_K(P)$.

Let $\beta = \text{br}_P^{\mathbb{E}K}: (\mathbb{E}K)^P \rightarrow \mathbb{E}L$ be the Brauer homomorphism. It induces a bijection between the primitive idempotents in $(\mathbb{E}K)^M = \mathbf{Z}(\mathbb{E}K)^P$ with defect P , and the primitive idempotents of $(\mathbb{E}L)^{\mathbf{N}_M(P)}$ with defect P . Since $\mathbf{N}_M(P) = LP$, the last idempotents are the block idempotents of L with defect group 1.

In particular, to ϑ corresponds a character $\varphi \in \text{Irr } L$ of defect zero. (When K is a p' -group, then this is the Glauberman correspondent of ϑ . See also [23, §5.12].) The correspondence commutes with Galois automorphisms. Thus we are in the situation of Hypothesis 9.2 with $\mathbb{K} = \mathbb{Q}_p$, the field of p -adic numbers, and $\mathbb{F} = \mathbb{F}_p$, the prime field with p elements. Let $e = \bar{e}_{(\vartheta, \mathbb{Q}_p)}$ and $f = \bar{e}_{(\varphi, \mathbb{Q}_p)}$ be the central primitive idempotents of $\mathbb{F}_p K$ and $\mathbb{F}_p L$ corresponding to the blocks of ϑ and φ over the prime field \mathbb{F}_p . (Remember that $\text{Gal}(\mathbb{Q}_p(\vartheta)/\mathbb{Q}_p) \cong \text{Gal}(\mathbb{E}/\mathbb{F}_p)$ canonically in this situation.) In this section, we set

$$i = \sum_{\gamma \in \text{Gal}(\mathbb{E}/\mathbb{F}_p)} (\bar{e}_\vartheta \bar{e}_\varphi)^\gamma.$$

11.1. Theorem. *In the situation just described, there is a magic crossed representation $\sigma: H/L \rightarrow (i\mathbb{F}_p K i)^L$. Moreover, σ can be chosen such that $\text{br}_P(\sigma(h)) = f$ for all $h \in \mathbf{C}_G(P)$.*

We list some corollaries.

11.2. Corollary. $\mathbb{F}_p Ge$ and $\mathbb{F}_p Hf$ are $(H/L\text{-graded})$ Morita equivalent. Any block ideal of $\mathbb{F}_p Ge$ is Morita equivalent to its Brauer correspondent with respect to P .

Proof. The first assertion follows from Corollary 6.12.

Now set $S = (i\mathbb{F}_p Ki)^L$ and remember that the magic crossed representation σ (with respect to $\mathbf{M}_n(\mathbb{F}_p) \cong S_0 \subseteq S$) yields an isomorphism

$$\kappa: \mathbb{F}_p Hf \rightarrow \mathbf{C}_{i\mathbb{F}_p Gi}(S_0), \quad h \mapsto h\sigma(h)^{-1}.$$

This restricts to an isomorphism from $\mathbf{Z}(\mathbb{F}_p Hf)$ to $\mathbf{Z}(i\mathbb{F}_p Gi)$. Furthermore, the map $z \mapsto zi$ is an isomorphism between $\mathbf{Z}(\mathbb{F}_p Ge)$ and $\mathbf{Z}(i\mathbb{F}_p Gi)$.

$$\mathbf{Z}(\mathbb{F}_p Ge) \xrightarrow{\cdot i} \mathbf{Z}(i\mathbb{F}_p Gi) \xleftarrow{\kappa} \mathbf{Z}(\mathbb{F}_p Hf).$$

These isomorphisms yield a bijection between the block idempotents of $\mathbb{F}_p Ge$ and $\mathbb{F}_p Hf$. Let $b \in \mathbf{Z}(\mathbb{F}_p Ge)$ and $c \in \mathbf{Z}(\mathbb{F}_p Hf)$ be block idempotents that correspond under these isomorphisms, that is, we have $bi = c\kappa$. The Morita equivalence that comes from the magic crossed representation σ then yields a Morita equivalence between the blocks $\mathbb{F}_p Gb$ and $\mathbb{F}_p Hc$. Thus we need to show that c is the Brauer correspondent of b for an appropriate choice of σ . An appropriate choice is any σ such that $\text{br}_P(\sigma(h)) = f$ for $h \in \mathbf{C}_G(P)$.

Write $c = \sum_{h \in H} c_h h$ and observe that $c_h = 0$ for $h \notin \mathbf{C}_H(P)$ [23, Theorem 5.2.8(ii)], since P is a normal p -subgroup of H . We compute

$$\begin{aligned} \text{br}_P(b) &= \text{br}_P(be) = \text{br}_P(b)f = \text{br}_P(bi) = \text{br}_P(c\kappa) \\ &= \text{br}_P\left(\sum_{h \in \mathbf{C}_G(P)} c_h h \sigma(h)^{-1}\right) \\ &= \sum_{h \in \mathbf{C}_G(P)} c_h h \text{br}_P(\sigma(h)^{-1}) \\ &= \sum_{h \in \mathbf{C}_G(P)} c_h h \cdot f = c, \end{aligned}$$

as was to be shown. \square

11.3. Corollary. *There is a bijection ι between $\text{Irr}(G \mid e_{(\vartheta, \mathbb{Q}_p)})$ and $\text{Irr}(H \mid e_{(\varphi, \mathbb{Q}_p)})$ with the properties given in Theorem 4.3. In particular, the bijection ι commutes with field automorphisms over \mathbb{Q}_p and preserves Schur indices over \mathbb{Q}_p . Corresponding characters lie in Brauer corresponding blocks (with respect to P).*

This corollary is a slight generalization of one of the main results of Turull's paper [30]. In fact, this section gives a somewhat different proof of Turull's result, but not completely independent. In particular, we need a fact about endopermutation modules depending on Dade's classification of these objects [4, 5]. This fact was announced by Dade [6], but without proof. A rather sketchy proof was published by Puig [24]. We use the complete exposition with detailed proofs given by Turull [30, Section 3].

Our aim in this section is to show that Turull's result can be derived naturally from the theory developed here. We do not need Turull's theory of the Brauer-Clifford group [31]. Also note that Turull assumes that K is a p' -group, and that Properties (j) and (k) in Theorem 4.3 and the connection with the Brauer correspondence give additional information.

Proof of Corollary 11.3. We will show below (Lemma 11.9) that

$$\dim_{\mathbb{E}}(i\mathbb{F}_p Ki)^L \equiv 1 \pmod{p}.$$

Thus $n = (\vartheta_L, \varphi) \equiv \pm 1 \pmod{p}$. (Of course, this is a well known property of the Glauberman correspondence, cf. [23, §5.12].) By Theorem 9.5, the magic crossed representation σ lifts to a magic representation in characteristic zero. The result follows from Theorem 6.13. \square

To prove Theorem 11.1, we need some facts about endopermutation modules and Dade P -algebras.

For the moment, let P be a p -group and \mathbb{E} an arbitrary field of characteristic p . A permutation $\mathbb{E}P$ -module is a module with an \mathbb{E} -basis that is permuted by P . (Such a basis is called P -stable.) A permutation P -algebra is a P -algebra A that has a P -stable basis. An $\mathbb{E}P$ -module V such that $\text{End}_{\mathbb{E}} V$ is a permutation P -algebra, is called an endopermutation module. A Dade P -algebra over \mathbb{E} is a permutation P -algebra S that is central simple over \mathbb{E} , and such that $S(P) \neq 0$. Equivalently, the central simple algebra S has a P -stable basis B containing 1_S . This means that S viewed as an $\mathbb{E}P$ -module is a permutation module containing the trivial module \mathbb{E} as direct summand.

We need some properties of permutation modules and Dade P -algebras. First some elementary facts:

11.4. Lemma.

- (a) *Every direct summand of a permutation P -module is itself a permutation P -module.*
- (b) *If A and B are permutation P -algebras, then*

$$(A \otimes_{\mathbb{E}} B)(P) \cong A(P) \otimes_{\mathbb{E}} B(P)$$

canonically.

- (c) *If S is a Dade P -algebra and $S \cong \mathbf{M}_n(\mathbb{E})$, then $S(P)$ is also a matrix ring over \mathbb{E} .*
- (d) *If P acts on $\mathbf{M}_n(\mathbb{E})$, where \mathbb{E} is a perfect field, then there is a unique group homomorphism $\sigma: P \rightarrow \text{GL}_n(\mathbb{E})$ such that $s^p = s^{\sigma(p)}$ for all $s \in \mathbf{M}_n(\mathbb{E})$.*

Proof. [29, Corollary 27.2, Proposition 28.3, Theorem 28.6(a), Corollary 21.4]. \square

The proofs of the next results depend on Dade's classification of endopermutation modules for abelian p -groups [4, 5].

11.5. Lemma. *Let P be an abelian p -group, let $\mathbb{F} \leq \mathbb{E}$ be finite fields and V an endopermutation $\mathbb{E}P$ -module. Then there is an endopermutation $\mathbb{F}P$ -module V_0 such that $V \cong V_0 \otimes_{\mathbb{F}} \mathbb{E}$.*

Proof. [30, Theorem 3.3]. \square

11.6. Corollary. *Let P be an abelian p -group, let $\mathbb{F} \leq \mathbb{E}$ be finite fields and V an endopermutation $\mathbb{E}P$ -module. Write $S = \text{End}_{\mathbb{E}} V \subseteq T = \text{End}_{\mathbb{F}} V$. Then S and T have P -stable bases, $S(P)$ embeds naturally in $T(P)$ and $\mathbf{C}_T(S) \cong \mathbf{C}_{T(P)}(S(P)) \cong \mathbb{E}$ naturally.*

That T has a P -stable basis means that V is an endopermutation module for P over \mathbb{F} , this is Corollary 3.4 in [30].

Proof of Corollary 11.6. By Lemma 11.5, there is an endopermutation module $V_0 \leq V_{\mathbb{F}P}$ such that $V \cong V_0 \otimes_{\mathbb{F}} \mathbb{E}$. Set

$$S_0 = \{s \in T \mid V_0 s \subseteq V_0\} \cong \text{End}_{\mathbb{F}} V_0.$$

Since V_0 is an endopermutation module, S_0 has a P -stable basis, B .

Set $Y = \text{End}_{\mathbb{F}} \mathbb{E}$. Then $T \cong S_0 \otimes_{\mathbb{F}} Y$ as \mathbb{F} -algebra. If we let P act trivially on Y , this is actually an isomorphism of P -algebras. Taking any \mathbb{F} -basis C of Y , we see that $\{b \otimes c \mid b \in B, c \in C\}$ is a P -stable basis of T .

It is clear that $\mathbf{C}_T(S) = \text{End}_S V \cong \mathbb{E}$. Let $\lambda(\mathbb{E}) = \text{End}_{\mathbb{E}} \mathbb{E} \subseteq Y$ and note that $\lambda(\mathbb{E}) \cong \mathbb{E}$ via multiplication. We have $\mathbf{C}_Y(\lambda(\mathbb{E})) = \lambda(\mathbb{E})$. The isomorphism $T \cong S_0 \otimes Y$ identifies S with $S_0 \otimes \lambda(\mathbb{E})$.

As $Y(P) = Y$ and $\mathbb{E}(P) = \mathbb{E}$, it now follows that $T(P) \cong S_0(P) \otimes_{\mathbb{F}} Y$ and $S(P) \cong S_0(P) \otimes_{\mathbb{F}} \mathbb{E}$ canonically. Thus also $\mathbf{C}_{T(P)}(S(P)) \cong \mathbb{E}$. \square

If S is a Dade P -algebra, then, by Lemma 11.4, Part (d), there is a unique group homomorphism $\sigma: P \rightarrow S$ inducing the action of P on S . Thus the notation $\mathbf{N}_{S^*}(P)$ is unambiguous, although strictly speaking we should write $\mathbf{N}_{S^*}(\sigma(P))$.

11.7. Lemma. *Let P be an abelian p -group, let S be a Dade P -algebra and let $\text{br}_P: S^P \rightarrow S(P)$ be the Brauer homomorphism. Then there is a group homomorphism $\varphi: \mathbf{N}_{S^*}(P) \rightarrow S(P)^*$ such that φ extends br_P and $\text{br}_P(c)^{\varphi(s)} = \text{br}_P(c^s)$ for all $s \in \mathbf{N}_{S^*}(P)$ and $c \in S^P$.*

Proof. [24] or [30, Theorem 3.11]. \square

Set

$$A = \{a \in \text{Aut } S \mid \sigma(P)^a = \sigma(P)\}.$$

Here $\text{Aut } S$ denotes the set of all ring automorphisms of S , not just the \mathbb{E} -algebra automorphisms. Since A stabilizes P , it follows that A acts naturally on $S(P)$. Lemma 11.7 can be strengthened:

11.8. Lemma. *If $\mathbb{E} = \mathbf{Z}(S)$ is finite in the situation of Lemma 11.7, the homomorphism $\varphi: \mathbf{N}_{S^*}(P) \rightarrow S(P)$ can be chosen such that $\varphi(s^a) = \varphi(s)^a$ for all $s \in \mathbf{N}_{S^*}(P)$ and $a \in A$.*

Proof. Let V_S be a simple S -module. Then V is an endopermutation module for P over \mathbb{E} . It is still an endopermutation module when viewed as a module over the prime field \mathbb{F}_p , by Corollary 11.6. Let $T = \text{End}_{\mathbb{F}_p} V$ and view $S \cong \text{End}_{\mathbb{E}} V$ as subset of T . Then $\mathbf{C}_T(S) = \mathbb{E}$ and $\mathbf{C}_T(\mathbb{E}) = S$.

The inclusion $S^P \subseteq T^P$ induces an injection $S(P) \hookrightarrow T(P)$ by Corollary 11.6, and $\mathbf{C}_{T(P)}(S(P)) = \mathbb{E}$.

By Lemma 11.7 applied to T , there is a homomorphism $\varphi: \mathbf{N}_{T^*}(P) \rightarrow T(P)$. We claim that $\varphi(\mathbf{N}_{S^*}(P)) \subseteq S(P)$. Let $s \in \mathbf{N}_{S^*}(P)$ and $z \in \mathbf{Z}(S)$. Then $\text{br}_P(z)^{\varphi(s)} = \text{br}_P(z^s) = \text{br}_P(z)$, so $\varphi(s)$ centralizes $\text{br}_P(\mathbf{Z}(S)) \cong \mathbf{Z}(S)$. Since $\mathbf{C}_{T(P)}(\mathbf{Z}(S)) = S(P)$, the claim follows.

Now let $a \in A$. By the Skolem-Noether theorem [9, p. 3.14], there is $t \in T^*$ such that $s^a = s^t$ for all $s \in S$. Since $P^a = P$, it follows that $t \in \mathbf{N}_{T^*}(P)$. Thus for $s \in \mathbf{N}_{S^*}(P)$,

$$\varphi(s^a) = \varphi(s^t) = \varphi(s)^{\varphi(t)}.$$

Now $s \in \mathbf{N}_{S^*}(P)$ given, there is $c \in S^P \subseteq T^P$ such that $\varphi(s) = \text{br}_P(c)$, as $\text{br}_P: S^P \rightarrow S(P)$ is surjective. Then by Lemma 11.7 applied to T we have

$$\varphi(s)^{\varphi(t)} = \text{br}_P(c)^{\varphi(t)} = \text{br}_P(c^t) = \text{br}_P(c^a) = \text{br}_P(c)^a = \varphi(s)^a,$$

where the second last equation follows from the definition of the action of A on $S(P)$. \square

We work now in the situation of Theorem 11.1. We use the notation introduced at the beginning of the section before the statement of Theorem 11.1. We set $S = (i\mathbb{F}_p Ki)^L$ and $Z = \mathbf{Z}(S) \cong \mathbb{E}$. We begin with a lemma.

11.9. Lemma. *S is a Dade P -algebra with $S(P) \cong \mathbb{E}$. (Thus $\dim_{\mathbb{E}} S \equiv 1 \pmod{p}$.) We may identify br_P^S with the restriction of $\text{br}_P^{\mathbb{F}K}$ to S .*

Proof. $\mathbb{F}K$ has a basis that is permuted by P . The same is thus true for the direct summand $i\mathbb{F}Ki$. By assumption, P centralizes $Z \cong \mathbf{Z}(\mathbb{F}Ke)$. Let B be some Z -basis of $\mathbb{F}Lf$. From $i\mathbb{F}Ki \cong S \otimes_Z \mathbb{F}Lf$ it follows that $i\mathbb{F}Ki \cong \bigoplus_{b \in B} S \otimes b$ as $\mathbb{F}P$ -module (remember that $L = \mathbf{C}_K(P)$). Thus $S \cong S \otimes b$ is a direct summand of $i\mathbb{F}Ki$ and thus a permutation P -module.

Let $\beta = \text{br}_P^{\mathbb{F}K} : (\mathbb{F}K)^P \rightarrow \mathbb{F}L$. Note that $\beta(i) = \beta(e) = f^2 = f$ and that $\beta(S^P) \subseteq \mathbf{Z}(\mathbb{F}Lf)$, as L centralizes S . It follows $S^P / (S^P \cap \ker \beta) \cong \beta(S^P) = \mathbf{Z}(\mathbb{F}Lf) \cong \mathbb{E}$. It remains to show that $S^P \cap \ker \beta = \ker \text{br}_P^S$. It is clear that

$$\ker \text{br}_P^S = \sum_{Q < P} S_Q^P \subseteq \sum_{Q < P} (\mathbb{F}K)_Q^P = \ker \beta.$$

Conversely, suppose

$$s = \sum_{Q < P} T_Q^P(a_Q) \in S^P \cap \ker \beta \quad \text{with } a_Q \in (\mathbb{F}K)^Q.$$

Since $s = isi = \sum_{Q < P} T_Q^P(ia_Qi)$, we may assume $a_Q \in (i\mathbb{F}Ki)^Q$. Now S is a direct summand of $i\mathbb{F}Ki$ as P -module. Letting $\pi : i\mathbb{F}Ki \rightarrow S$ be the corresponding projection, we see

$$s = \pi(s) = \sum_{Q < P} T_Q^P(\pi(a_Q)) \in \sum_{Q < P} S_Q^P = \ker \text{br}_P^S$$

as was to be shown. The proof is finished. \square

Proof of Theorem 11.1. Suppose we have a counterexample to the theorem with $|K/L|$ minimal. Then clearly $L < K$. Set $C = \mathbf{C}_P(K)$, so that $C < P$. Let P_0/C be a chief factor of H/C with $P_0 \leq P$ and set $L_0 = \mathbf{C}_K(P_0)$. As $P_0 > C$, we have $L_0 < K$. Now the composition

$$(\mathbb{E}K)^P \xrightarrow{\text{br}_{P_0}} (\mathbb{E}L_0)^{P/P_0} \xrightarrow{\text{br}_{P/P_0}} \mathbb{E}L$$

gives just $\beta = \text{br}_P$. Let $\eta \in \text{Irr } L_0$ be defined by $\text{br}_{P_0}(\overline{e_\eta}) = \overline{e_\eta}$. Set $M_0 = KP_0$ and $H_0 = \mathbf{N}_G(P_0)$. Note that $M_0 \trianglelefteq G$. We have decomposed the configuration $(G, H, K, L, \vartheta, \varphi)$ into two configurations: The configuration $(G, H_0, K, L_0, \vartheta, \eta)$ with M_0 and P_0 instead of M and P , and the configuration $(H_0, H, L_0, L, \eta, \varphi)$ with PL_0 and P instead of M and P (see Figure 4).

If $L < L_0$, then induction applies and yields magic crossed representations σ_1 and σ_2 for the two configurations $(G, H_0, K, L_0, \vartheta, \eta)$ and $(H_0, H, L_0, L, \eta, \varphi)$. Now note that all the hypotheses made in Section 8 are met. We apply Proposition 8.5 to conclude that there is a magic crossed representation $\sigma : H \rightarrow S$, such that

$$\sigma(h)j = \sigma_1(h)\sigma_2(h) \quad \text{with } j = \sum_{\gamma} (\overline{e_\vartheta e_\eta e_\varphi})^\gamma.$$

By induction, we may also assume that $\text{br}_{P_0}(\sigma_1(h)) = \overline{e_{(\eta, \mathbb{Q}_p)}}$ for $h \in \mathbf{C}_G(P_0)$ and $\text{br}_P(\sigma_2(h)) = \overline{e_{(\varphi, \mathbb{Q}_p)}} = f$ for $h \in \mathbf{C}_G(P)$. As $\text{br}_P(\overline{e_\vartheta}) = \text{br}_P(\overline{e_\eta}) = \overline{e_\varphi}$, it follows that $\text{br}_P(j) = f$. Thus for $h \in \mathbf{C}_G(P)$,

$$\begin{aligned} \text{br}_P(\sigma(h)) &= \text{br}_P(\sigma(h)j) = \text{br}_P(\text{br}_{P_0}(\sigma(h)j)) \\ &= \text{br}_P(\text{br}_{P_0}(\sigma_1(h)\sigma_2(h))) \\ &= \text{br}_P(\overline{e_{(\eta, \mathbb{Q}_p)}}\sigma_2(h)) \\ &= f. \end{aligned}$$

It follows that our configuration is not a counterexample.

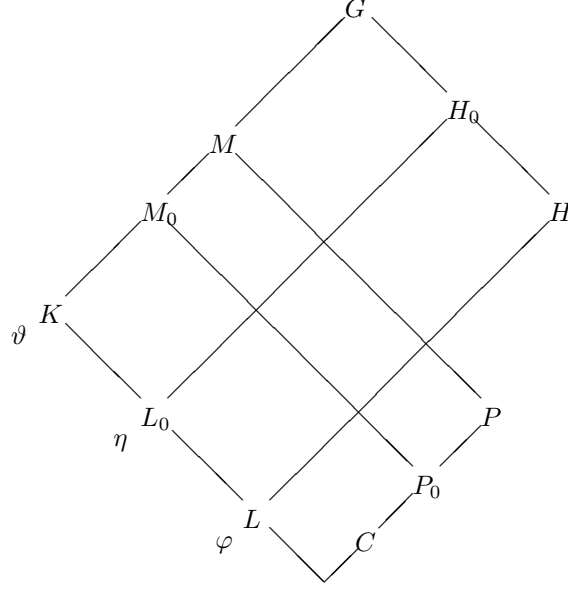


FIGURE 4. Proof of Theorem 11.1

Thus we can assume $L = L_0$. We may replace P by P_0 and assume that P/C is a chief factor of H . In particular, we can assume that P/C is abelian.

By Lemma 11.4, Part (d), there is a unique $\sigma: P/C \rightarrow S$ which induces the action of P on S . Let V be a simple S -module, which becomes an endopermutation $\mathbb{E}P$ -module via σ . By Lemma 11.5, there is an $\mathbb{F}P$ -module V_0 such that $V \cong V_0 \otimes_{\mathbb{F}} \mathbb{E}$. We may assume $V_0 \leq V$. Let $S_0 = \{s \in S \mid V_0 s \subseteq V_0\}$. Then $\sigma(P) \subseteq S_0$ and $S_0 \cong \mathbf{M}_n(\mathbb{F})$.

The action of H/L on S yields a projective crossed representation $\tilde{\sigma}: H/L \rightarrow S^*$, that is, for each $h \in H$ there is an element $\tilde{\sigma}(h) = \tilde{\sigma}(Lh) \in S^*$ such that $s^h = s^{\tilde{\sigma}(h)}$ for all $s \in S_0$, that is, $\tilde{\sigma}$ is crossed with respect to S_0 . (Note that we may assume $\tilde{\sigma}(x) = \sigma(x)$ for $x \in P$.)

Let $x \in P$ and $h \in H$. By the uniqueness of σ on P , we must have $\sigma(x)^h = \sigma(x^h)$. It follows that

$$\sigma(x)^{\tilde{\sigma}(h)} = \sigma(x)^h = \sigma(x^h) \in \sigma(P).$$

Thus $\tilde{\sigma}(h) \in \mathbf{N}_{S^*}(\sigma(P))$.

Let $\varepsilon: H/L \rightarrow \text{Aut } S$ be defined by $(s_0 z)^{\varepsilon(h)} = s_0 \cdot z^h$ for $s_0 \in S_0$ and $z \in \mathbf{Z}(S) \cong \mathbb{E}$. To $\tilde{\sigma}$ belongs a cocycle $\alpha \in Z^2(H/L, \mathbf{Z}(S)^*)$ such that

$$\tilde{\sigma}(x)^{\varepsilon(y)} \tilde{\sigma}(y) = \alpha(x, y) \tilde{\sigma}(xy) \quad \text{for all } x, y \in H/L.$$

Now we apply the homomorphism $\varphi: \mathbf{N}_{S^*}(\sigma(P)) \rightarrow S(P)^* \cong \mathbb{E}^*$ of Lemmas 11.7 and 11.8 to this equation. It follows that

$$\varphi(\tilde{\sigma}(x))^{\varepsilon(y)} \varphi(\tilde{\sigma}(y)) = \alpha(x, y) \varphi(\tilde{\sigma}(xy)).$$

(Here, $\varphi(\tilde{\sigma}(x)^{\varepsilon(y)}) = \varphi(\tilde{\sigma}(h))^{\varepsilon(y)}$ follows from Lemma 11.8.) Thus

$$\sigma(h) := \varphi(\tilde{\sigma}(h))^{-1} \tilde{\sigma}(h)$$

defines a magic crossed representation $\sigma: H/L \rightarrow S$. Moreover, for this choice of σ we have $\varphi(\sigma(h)) = 1_{S(P)}$. Here, we may identify $S(P)$ with $\mathbf{Z}(\mathbb{F}L f)$. Since φ extends br_P , it follows that for $c \in \mathbf{C}_G(P)$ we have $\text{br}_P(\sigma(c)) = \varphi(\sigma(c)) = f$. The proof is finished. \square

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